

Mathematical Principles of Dynamic Systems and the Foundations of Quantum Physics

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Abstract

Everybody agrees that quantum physics is strange, and that the world view it implies is elusive. However, it is rarely considered that the theory might be opaque because the mathematical language it employs is inarticulate. Perhaps, if a mathematical language were constructed specifically to handle the theory's subject matter, the theory itself would be clarified. This article explores that possibility. It presents a simple but rigorous language for the description of dynamics, experiments, and experimental probabilities. This language is then used to answer a compelling question: What is the set of allowed experiments? If an experiment is allowed, then the sum of the probabilities of its outcomes must equal 1. If probabilities are non-additive, there are necessarily sets of outcomes whose total probability is not equal to 1. Such experiments are therefore not allowed. That being the case, in quantum physics, which experiments are allowed, and why are the rest disallowed? What prevents scientists from performing the disallowed experiments? By phrasing these questions within our mathematical language, we will uncover answers that are complete, conceptually simple, and clearly correct. This entails no magic or sleight of hand. To write a rigorous mathematical language, all unnecessary assumptions must be shed. In this way, the thicket of ad hoc assumptions that surrounds quantum physics will be cleared. Further, in developing the theory, the logical consequences of the necessary assumptions will be laid bare. Therefore, when a question can be phrased in such a language, one can reasonably expect a clear, simple answer. In this way we will dispel much of the mystery surrounding quantum measurements, and begin to understand why quantum probabilities have their peculiar representation as products of Hilbert space projection operators.

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I. INTRODUCTION

Our fascination with quantum physics has as much to do with its strangeness as its success. This strangeness can conjure contradictory responses: on the one hand, the sense that science has dug so deep as to touch upon profound metaphysical questions, and on the other, the sense that something is amiss, as science should strive to uncover simple explanations for seemingly strange phenomena. Fans of the first response will find little of interest in the paper, for it explores the second.

Let's start by noting that the mathematical language employed by quantum mechanics was not developed to investigate the types of problems that are of interest in that field. Hilbert spaces, for example, were developed to investigate analogies between certain function spaces & Euclidean spaces; they were only later adopted by physicists to describe quantum systems. This is in sharp contrast with classical mechanics; the development of differential calculus was, to a large extent, driven by the desire to describe the observed motion of bodies - the very question with which classical mechanics is concerned. In consequence, it is no exaggeration to say that if a question is well defined within classical mechanics, it can be described using calculus.

In quantum mechanics, the mathematical language is far less articulate. For example, it leaves unclear what empirical properties a system must possess in order for the quantum description to apply. It's also unclear how, and even whether, the language of quantum mechanics can be used to describe the experiments employed to test the theory. It is similarly unclear whether and/or how quantum mechanics can be used to describe the world of our direct experience.

Such considerations lead to a simple question: To what extent are the difficulties of quantum theory due to limitations in our ability to phrase relevant questions in the theory's mathematical language? If such limitations do play a role, it would not represent a unique state of affairs. As most everyone knows, Zeno's paradoxes seemed to challenge some of our most basic notions of time and motion, until calculus resolved the paradoxes by creating a clear understanding of the continuum. Somewhat more remotely, the drawing up of annual calendars (and other activities founded on cyclic heavenly activity) was once imbued with a mystery well beyond our current awe of quantum physics. With the slow advance of the theory of numbers the mystery waned, until now such activity requires nothing more

mysterious than straightforward arithmetic.

In this article, the question of whether a similar situation exists for quantum mechanics will be investigated. Three simple mathematical theories will be created, each addressing a basic aspect of dynamic systems. The resulting mathematical language will then be used to analyze quantum systems. If the analysis yields core characteristics of quantum theory, some portion of the theory's underpinnings will necessarily be revealed, and a measure of insight into previously mentioned questions ought to be gained.

The mathematics will be constructed to speak to two of the most basic differences between experimental results in quantum mechanics and those in classical mechanics: in quantum mechanics experimental outcomes are non-deterministic & the experimental probabilities are not additive. Non-additivity in turn puts limitations on the kinds of experiments that can be performed. To see this, note that if an experiment has outcomes of $\{X, \neg X\}$, and another has outcomes of $\{X_1, X_2, \neg X\}$ then $P(X) + P(\neg X) = 1 = P(X_1) + P(X_2) + P(\neg X)$, and so $P(X) = P(X_1) + P(X_2)$; if $P(X) \neq P(X_1) + P(X_2)$ then one of these experiments can not be performed (the only other possibility is that the probability of a given outcome depends on the make-up of the experiment as a whole, but this is not the case in quantum physics). Turning this around, if there is no set Y , s.t. $\{X\} \cup Y$ and $\{X_1, X_2\} \cup Y$ are both sets of experimental outcomes, one may wonder if we can expect $P(X)$ to equal $P(X_1) + P(X_2)$.

This line of reasoning provides some of the central questions to be answered in this article: What sorts of experiments can be performed? What limits scientists to only being able to perform these experiments? What does the set of allowed experiments imply about the nature of the experimental probabilities? How do these experimental probabilities correspond to quantum probabilities? We will seek simple, readily understandable answers to these questions.

Of the three mathematical theories to be constructed, the first will be a simple theory of dynamic systems that encompasses both deterministic and non-deterministic dynamics. Determinism refers to the particular case in which a complete knowledge of the present grants complete knowledge of the future; all other cases represent types of non-determinism. The theory of dynamic systems will then be utilized to construct a theory of experiments; dynamic systems will be used to describe both experiments as a whole, and the (sub)systems whose natures the experiments probe. The analysis will be somewhat analogous to that found in automata theory: when a (sub)system path is “read into” an experimental set-up,

the set-up determines which outcome the path belongs to. By understanding how such processes can be constructed, we can obtain an understanding of what types of experiments are performable, reproducible, and have well defined outcomes.[1]

Finally, building on this understanding, a probability theory will be given for collections of experiments by assuming that the usual rules of probability and statistics hold on the set of outcomes for any individual experiment & that if any two experiments in the collection share an outcome, then they agree on that outcome's probability.

These constructs will then be applied to quantum measurements, which are commonly described in terms of projection operators in a Hilbert space. It will be found that the nature and structure of these measurements can indeed be reduced to a clear, simple, rational understanding. At no point will there be any need to invoke anything the least bit strange, spooky, or beyond the realm of human understanding, nor any need to rely on any procedures that are utilized because they work even though we don't understand why.

Because this paper only considers a fraction of all quantum phenomenon, no claim can be made that all (or even most) quantum phenomenon can yield to some simple, rational understanding. What we seek to establish is more modest - that at least some of our bafflement in the face of quantum physics is due to the manner with which we address the phenomenon, rather than the nature of the phenomena themselves.

On the Question of Interpretations

Though quantum interpretations is a large topic, in this article it will play a small role. However, because a theory can not be fully understood without some notion of its possible interpretations, a word or two are in order before proceeding.

The concept of an "interpretation" will be taken here as being more or less equivalent to the mathematical concept of a "model".[2] The theories in this article will have many models, and there will be no attempt to single out any one as being preferred. In this section, informal sample models will be given for the two properties of central interest: non-deterministic dynamics and non-additive probabilities. These should help provide a background understanding for the theories.

Two Models for Non-Determinism

As noted previously, non-determinism simply means that a complete knowledge of a system's current state does not imply a complete knowledge of all the system's future states. In the simplest model of non-determinism, at any given time the system is in a single state, but that state doesn't contain enough information to be able to deduce what all the future states will be. This will be referred to as *type-i* non-determinism (the "i" standing for "individual", because the system is always in an individual state). Turning towards the past rather than the future, this is akin to the situations found in archeology and paleontology, in which greater knowledge of the present state does yield greater knowledge of the past, but you would not expect any amount of knowledge of the present to yield a complete knowledge of all history.

In *type-m* non-determinism, the system takes multiple paths simultaneously - every path it can take, it will take (the "m" in "type-m" stands for "multiple"). Quantum mechanics is often interpreted as displaying type-m non-determinism; for example, in a double slit experiment, the particle is viewed as traveling through both slits.

There are also mixed models. In non-deterministic automata, the input shows type-i non-determinism (in that a single character in the input string will not determine what the rest of the input must be), while the automata has type-m non-determinism (as individual characters are read in, the automata "non-deterministically samples" all possible transitions). Some quantum mechanical interpretations invert that view - the system being experimented on is seen as having type-m non-determinism while the experimental set-up is seen as showing type-i non-determinism (it's often further assumed that if the system were to be deterministic, then the experimental set-up would also be deterministic, and more specifically, classical). In a similar manner, decoherence often entails an unspoken assumption that a system displays type-m non-determinism if the non-diagonal elements of the density matrix do not vanish, and displays type-i non-determinism otherwise; in this sense, the non-determinism is considered to be type-i in so far the probabilities are additive, and type-m in so far as the probabilities are non-additive.

It's important to stress that type-i, type-m, and mixed models are not the only possible models for non-determinism; many-worlds interpretations, for example, provide yet another type of model.

Two Models for Non-Additivity

Two models will now be given for systems that display both non-determinism & non-additive probabilities.

Systems with type-i non-determinism can have non-additive probabilities if interactions with the measuring devices can not be made arbitrarily small (e.g., if the fields mediating the interactions are quantized). As an example, imagine experimental outcome X and outcomes $\{X_1, X_2\}$ s.t $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$. The minimal interactions required to determine X will in general be different from the minimal interactions to determine X_1 or X_2 . These differing interactions will cause different deflections to the system paths, which can result in differences between the statistical likelihood of the outcome being X and the statistical likelihood of the outcome being X_1 or X_2 . Such a state of affairs can be referred to as the “intuitive model”, because it allows our physical intuition to be applied.

For type-m non-determinism, one manner in which non-additive probabilities can appear is if the various paths that the system takes interfere with one other. In this case, for outcome X , all paths corresponding to X may interfere, whereas for $\{X_1, X_2\}$ paths can only interfere if they correspond to the same X_i ; the differences in interference then lead to different probabilities for the outcomes. This can be referred to as the “orthodox model”, because it is shared by many of the most widely accepted interpretations of quantum mechanics.

Once again, these are just two “sample” models for the non-additivity; there are many others.

II. DYNAMICS

To begin, we require a theory of dynamic systems. Existing theories tend to make assumptions that are violated by quantum systems, so here a simple, lightweight theory will be presented; a theory in which all excess assumptions will be stripped away.

A. Parameters

Dynamic systems are systems that can change over time; they are parametrized by time. In a sense, the one feature that all dynamic systems have in common is that they are parametrized. We therefore begin by quickly reviewing the concept of a parameter.

Parameters come in variety of forms. For example, some systems have discrete parameters, while others have continuous parameters. None the less, all parameters share several basic features. Specifically, a parameter must be totally ordered, and must support addition. Thus, parameters are structures with signature $(\Lambda, <, +, 0)$, where $<$ is a total ordering, $+$ is the usual addition function, and 0 is the additive identity. It is shown in Appendix D that this, together with the requirement that there be an element greater than 0 , yields the general concept of a parameter. The set of Real numbers are parameters in this sense, as are the Integers, the Rationals, and all infinite ordinals.

This is, however, a little too general. First, in this article we will only be interested in parameters whose values are finite. Second, for reasons of analysis, we will only be interested in parameters that are Cauchy convergent. These two requirements are equivalent to adding a completeness axiom; this axiom states that all subsets of Λ that are bounded from above have a least upper bound. This limits the models to only four, classified by whether the parameter are discrete or continuous, and whether or not they are bounded from below by 0 . These four types of parameters are closely related to the canonical number systems, and can be readily constructed from them by introducing the parameter value $\mathbf{1}$, and multiplication by a number.

If the parameter is discrete, assign “ $\mathbf{1}$ ” to be the successor to 0 ; if it’s continuous, choose “ $\mathbf{1}$ ” to be any parameter value greater than 0 . The choice of $\mathbf{1}$ sets the scale (e.g., 1 second); it is because parameters have a scale that multiplication is not defined on parameters.

If Λ is a parameter and $\lambda \in \Lambda$, λ added to itself n times will be denoted $n\lambda$ (for example, $3\lambda \equiv \lambda + \lambda + \lambda$). $0\lambda \equiv 0$.

As shown in Appendix D, constructing the four types of parameters from numbers is now straightforward. For any discrete, bounded from below parameter, $(\Lambda, <, +, 0)$, the following will hold: $\Lambda = \{n\mathbf{1} : n \in \mathbb{N}\}$, $n\mathbf{1} + m\mathbf{1} = (n + m)\mathbf{1}$, and $n\mathbf{1} > m\mathbf{1}$ iff $n > m$. Discrete, unbounded parameters are similar, but with the Integers replacing the natural numbers. For continuous, unbounded parameters, take $\mathbf{1}$ to be any positive value, and replace the natural numbers with the Reals (multiplication by a real number is defined in Appendix D). Similarly, for continuous, bounded parameters, replace the natural numbers with the non-negative Reals. This close correspondence between parameters and numbers is why the two concepts are often treated interchangeably.

We conclude this section by reviewing some standard notation. First, the notation for

parameter intervals:

Definition 1. If Λ is a parameter and $\lambda_1, \lambda_2 \in \Lambda$

$$[\lambda_1, \lambda_2] = \{\lambda \in \Lambda : \lambda_1 \leq \lambda \leq \lambda_2\}$$

$$(\lambda_1, \lambda_2) = \{\lambda \in \Lambda : \lambda_1 < \lambda < \lambda_2\}$$

and similarly for $[\lambda_1, \lambda_2)$ and $(\lambda, \lambda_2]$.

$$[\lambda_1, \infty] = \{\lambda \in \Lambda : \lambda_1 \leq \lambda\}$$

$$[-\infty, \lambda_1] = \{\lambda \in \Lambda : \lambda \leq \lambda_1\}$$

and similarly for (λ, ∞) , etc. (note that $[\lambda_1, \infty) = [\lambda_1, \infty]$).

Next, functions for least upper bound and greatest lower bound:

Definition 2. If Λ is a parameter and $\chi \subset \Lambda$ then if χ is bounded from above, $\text{lub}(\chi)$ is χ 's least upper bound, and if χ is bounded from below, $\text{glb}(\chi)$ is χ 's greatest lower bound.

$\text{Min}(\chi)$ is equal to $\text{glb}(\chi)$ if χ is bounded from below, and $-\infty$ otherwise. Similarly, $\text{Max}(\chi)$ is equal to $\text{lub}(\chi)$ if χ is bounded from above, and ∞ otherwise.

And lastly, the definition of subtraction:

Definition 3. If Λ is a parameter and $\lambda, \lambda' \in \Lambda$:

$$\lambda - \lambda' \equiv \begin{cases} 0 & \text{if } \text{Min}(\Lambda) = 0 \text{ and } \lambda < \lambda' \\ \Delta\lambda & \text{where } \Delta\lambda + \lambda = \lambda', \text{ otherwise} \end{cases}$$

It follows from the numeric constructions outlined above that $\lambda - \lambda'$ always yields a unique element of Λ .

B. Parametrized Functions and Dynamic Sets

Parametrized functions and dynamic sets are the rudimentary concepts on which all else will be built. This section introduces them, along with their notational language. We start with parametrized functions.

Definition 4. A *parametrized function* is a function whose domain is a parameter.

If f is a parametrized function, and $[x_1, x_2]$ is an interval of $\text{Dom}(f)$, then $f[x_1, x_2]$ is f restricted to domain $[x_1, x_2]$ (values of $x_1 = -\infty$ and/or $x_2 = \infty$ are allowed.)

We define one operation on parametrized functions, concatenation.

Definition 5. If f and g are parametrized functions, $Dom(f) = Dom(g)$, and $f(\lambda) = g(\lambda)$, then $f[x_1, \lambda] \circ g[\lambda, x_2]$ is the function on domain $[x_1, x_2]$ s.t.

$$f[x_1, \lambda] \circ g[\lambda, x_2](\lambda') = \begin{cases} f(\lambda') & \text{if } \lambda' \in [x_1, \lambda] \\ g(\lambda') & \text{if } \lambda' \in [\lambda, x_2] \end{cases}$$

That is all that's needed for parametrized functions. They will now be used to define dynamic sets.

Definition 6. A *dynamic set*, S , is any non-empty set of parametrized functions s.t. all elements share the same parameter

If $f \in S$ then f can be referred to as a *path*, and will be generally be written \bar{p}

With $\bar{p} \in S$, $\Lambda_S \equiv Dom(\bar{p})$ (that is, Λ_S is the parameter that all elements of S share)

$\mathcal{P}_S \equiv \bigcup_{\bar{p} \in S} Ran(\bar{p})$; the elements of \mathcal{P}_S are *states*

$S[x_1, x_2] \equiv \{\bar{p}[x_1, x_2] : \bar{p} \in S\}$

For $\lambda \in \Lambda_S$, $S(\lambda) \equiv \{p \in \mathcal{P}_S : \text{for some } \bar{p} \in S, \bar{p}(\lambda) = p\}$ (That is, $S(\lambda)$ is the set of possible states at time λ)

$Uni(S) \equiv \{(\lambda, p) \in \Lambda_S \otimes \mathcal{P}_S : p \in S(\lambda)\}$ (“ $Uni(S)$ ” is the “universe” of S - the set of all possible time-state pairs)

In the above definition, “time” was occasionally mentioned. In what follows, we will refer only to “the parameter”, and not “time”. This is because “time” has grown into an overloaded concept. For example, in relativity the time measured on a clock is a function of the path the clock takes. Time measured by a clock is generally referred to as the “proper time”. On the other hand, in relativity theory a particle state is generally taken to be a four dimensional vector, \vec{x} , and coordinates are often chosen so that the “0th” element parametrizes the dynamics. This coordinate is generally referred to as the “time coordinate”, and this notion of time referred to as “coordinate time”. For such single particle systems, the elements of $Uni(S)$ will be of the form (λ, \vec{x}) , and for coordinates that have a “time coordinate”, the 0th element of \vec{x} will always equal λ . Since coordinates may be used to describe all paths, and “proper time” is path dependent, it follows that in such this case λ may not equal the proper time.

In this paper, all such complications will be shrugged off. First, we will make no attempt to map states onto coordinates. Moreover, there will be a preference to reserve the term “time” for the quantity that is measured by clocks. Since this may or may not equal the

quantity used to parametrize the system dynamics (depending on the path the clock takes), we retire the term “time” and speak only of “parameters”.

Back to dynamic sets.

The concatenation operation can be extended to sets of path-segments.

Definition 7. If S is a dynamic set and A and B are sets of partial paths of S then $A \circ B \equiv \{\bar{p}_1[x_1, \lambda] \circ \bar{p}_2[\lambda, x_2] : \bar{p}_1[x_1, \lambda] \in A, \bar{p}_2[\lambda, x_2] \in B \text{ and } \bar{p}_1(\lambda) = \bar{p}_2(\lambda)\}$

The following notation for various sets of path-segments will prove quite useful.

Definition 8. If S is a dynamic set, $(\lambda, p), (\lambda_1, p_1), (\lambda_2, p_2) \in Uni(S)$, and $\lambda_1 \leq \lambda_2$

$$S_{\rightarrow(\lambda, p)} \equiv \{\bar{p}[-\infty, \lambda] : \bar{p} \in S \text{ and } \bar{p}(\lambda) = p\}$$

$$S_{(\lambda, p) \rightarrow} \equiv \{\bar{p}[\lambda, \infty] : \bar{p} \in S \text{ and } \bar{p}(\lambda) = p\}$$

$$S_{(\lambda_1, p_1) \rightarrow (\lambda_2, p_2)} \equiv \{\bar{p}[\lambda_1, \lambda_2] : \bar{p} \in S, \bar{p}(\lambda_1) = p_1, \text{ and } \bar{p}(\lambda_2) = p_2\}$$

If S is a dynamic set and $p, p_1, p_2 \in \mathcal{P}_S$

$$S_{\rightarrow p} \equiv \{\bar{p}[-\infty, \lambda] : \bar{p} \in S, \lambda \in \Lambda_S, \text{ and } \bar{p}(\lambda) = p\}$$

$$S_{p \rightarrow} \equiv \{\bar{p}[\lambda, \infty] : \bar{p} \in S, \lambda \in \Lambda_S, \text{ and } \bar{p}(\lambda) = p\}$$

$$S_{p_1 \rightarrow p_2} \equiv \{\bar{p}[\lambda_1, \lambda_2] : \bar{p} \in S, \lambda_1, \lambda_2 \in \Lambda_S, \bar{p}(\lambda_1) = p_1, \text{ and } \bar{p}(\lambda_2) = p_2\}$$

If S is a dynamic set, $Y, Z \subset \mathcal{P}_S$

$$S_{Y \rightarrow Z} \equiv \bigcup_{p \in Y, p' \in Z} S_{p \rightarrow p'}$$

...and similarly for $S_{\rightarrow Z}, S_{Y \rightarrow}, S_{(\lambda, Y) \rightarrow (\lambda, Z)}$, etc.

This notation may be extended as needed. For example, $S_{\rightarrow p_1 \rightarrow (\lambda_2, p_2) \rightarrow p_3 \rightarrow}$ is the set of all paths in S that pass through p_1 , then through (λ_2, p_2) , then through p_3 .

One of the most basic properties of a dynamic systems is whether or not its dynamics can change over time. A dynamic set is homogeneous if its dynamics are the same regardless of when a state occurs. More formally:

Definition 9. A dynamic set, S , is *homogeneous* if for all $(\lambda_1, p), (\lambda_2, p) \in Uni(S)$, $\lambda_1 \leq \lambda_2$, $\bar{p} \in S_{\rightarrow(\lambda_1, p) \rightarrow}$ iff there exists a $\bar{p}' \in S_{\rightarrow(\lambda_2, p) \rightarrow}$ s.t. for all $\lambda \in \Lambda_S$, $\bar{p}(\lambda) = \bar{p}'(\lambda + \lambda_2 - \lambda_1)$.

Note that if Λ_D is bounded from below, S can still be homogeneous; it would simply mean that the paths running through (λ_1, p) and those running through (λ_2, p) only differ by a shift & by the fact the initial part of the paths running through (λ_1, p) are cut off.

A related property is whether or not a given state, or set of states, can occur at any time.

Definition 10. If S is a dynamic set, $p \in \mathcal{P}_S$ is *homogeneously realized* if for every $\lambda \in \Lambda_S$, $p \in S(\lambda)$.

$A \subset \mathcal{P}_S$ is *homogeneously realized* if every $p \in A$ is.

C. Dynamic Spaces

Dynamic sets embrace quite a broad concept of dynamics. This can make them difficult to work with, and so it is often helpful to make further assumptions about the system dynamics. A common assumption for closed systems is that the system's possible future paths are determined entirely by its current state. This notion is captured by the following type of dynamic set:

Definition 11. A *dynamic space*, D , is a dynamic set s.t. if $\bar{p}, \bar{p}' \in D$, $\lambda \in \Lambda_D$, and $\bar{p}(\lambda) = \bar{p}'(\lambda)$, then $\bar{p}[-\infty, \lambda] \circ \bar{p}'[\lambda, \infty] \in D$.

Thus, dynamic spaces are closed under concatenation. This will prove to be an enormous simplification. Closed systems are generally assumed to have this property, and so in what follows, closed systems will always be assumed to be dynamic spaces. (Note that a system being experimented on interacts with experimental equipment, and so is not closed. As a result, such a system might not be a dynamic space.)

Definition 12. If D is a dynamic space, $D_{\square_1 \rightarrow \square_2}$ may be written $\square_1 \rightarrow \square_2$. (For example, $D_{p_1 \rightarrow p_2}$ may be written $p_1 \rightarrow p_2$).

Similarly, $D_{\square_1 \rightarrow \square_2} \circ D_{\square_2 \rightarrow \square_3}$ may be written $\square_1 \rightarrow \square_2 \rightarrow \square_3$, $D_{\rightarrow \square_1} \circ D_{\square_1 \rightarrow \square_2} \circ D_{\square_2 \rightarrow}$ may be written $\rightarrow \square_1 \rightarrow \square_2 \rightarrow$, etc.

If D is a dynamic space then, for example, $D_{p_1 \rightarrow (\lambda_2, p_2) \rightarrow p_3} = D_{p_1 \rightarrow (\lambda_2, p_2)} \circ D_{(\lambda_2, p_2) \rightarrow p_3}$; if, on the other hand, S is simply a dynamic set, we could only assert that $S_{p_1 \rightarrow (\lambda_2, p_2) \rightarrow p_3} \subset S_{p_1 \rightarrow (\lambda_2, p_2)} \circ S_{(\lambda_2, p_2) \rightarrow p_3}$. This is why, for dynamic spaces, we may relax the notation and simply refer to $p_1 \rightarrow (\lambda_2, p_2) \rightarrow p_3$, while for dynamic sets we need to be clear about whether we mean $S_{p_1 \rightarrow (\lambda_2, p_2)} \circ S_{(\lambda_2, p_2) \rightarrow p_3}$ or $S_{p_1 \rightarrow (\lambda_2, p_2) \rightarrow p_3}$.

For dynamic spaces the use of outer arrows, such as $\rightarrow (\lambda_1, p_1) \rightarrow p_2 \rightarrow$, may be extended to arbitrary sets of partial paths:

Definition 13. If D is a dynamic space, and A is a set of partial paths s.t. if $\bar{p}[x_1, x_2] \in A$ then $\bar{p}[x_1, x_2] \in D[x_1, x_2]$,

$\bar{p} \in \rightarrow A \rightarrow$ if $\bar{p} \in D$ and for some $\bar{p}'[x_1, x_2] \in A$, $\bar{p}[x_1, x_2] = \bar{p}'[x_1, x_2]$.
 $\bar{p}[x, \infty] \in A \rightarrow$ if $\bar{p}[x, \infty] \in D[x, \infty]$ and for some $\bar{p}'[x, x_1] \in A$, $\bar{p}[x, x_1] = \bar{p}'[x, x_1]$.
 $\bar{p}[-\infty, x] \in \rightarrow A$ if $\bar{p}[-\infty, x] \in D[-\infty, x]$ and for some $\bar{p}'[x_0, x] \in A$, $\bar{p}[x_0, x] = \bar{p}'[x_0, x]$.

D. Special Sets

Although the notation introduced in Secs IIB and IIC will be sufficient for nearly all circumstances, two straightforward additions will prove useful in the discussion of experiments.

First, it will be useful to isolate the subset of $X \rightarrow Y$ consisting of the path-segments that don't re-enter X after their start; it will similarly be useful to isolate the path-segments that don't enter Y until their end.

Definition 14. If S is a dynamic set and $X, Y, Z \subset \mathcal{P}_S$:

$$S_{X \rightarrow Y|Z} \equiv \{\bar{p}[\lambda_1, \lambda_2] \in S_{X \rightarrow Y} : \text{for } \lambda \in [\lambda_1, \lambda_2), \bar{p}(\lambda) \notin Z\}$$

$$S_{Z|X \rightarrow Y} \equiv \{\bar{p}[\lambda_1, \lambda_2] \in S_{X \rightarrow Y} : \text{for } \lambda \in (\lambda, \lambda_2], \bar{p}(\lambda) \notin Z\}$$

$S_{X \rightarrow Y|Y}$ is then the set of path segments from S that start at X and end at Y , but do not enter Y before the end of the segment. Similarly for $S_{X|X \rightarrow Y}$.

$S_{X/Y}$, to be introduced momentarily, bears a resemblance to $S_{X \rightarrow Y|Y}$, but overcomes a difficulty that occurs in the continuum:

Definition 15. $S_{X/Y} \equiv \{\bar{p}[\lambda_1, \lambda_2] : \bar{p}[\lambda_1, \infty] \in S_{X \rightarrow}, Y \cap \text{Ran}(\bar{p}[\lambda_1, \infty]) \neq \emptyset, \text{ and } \lambda_2 = \text{glb}(\bar{p}^{-1}[\lambda_1, \infty][Y])\}$

In the above definition, λ_2 is either the first time Y occurs in $\bar{p}[\lambda_1, \infty]$ or, if the "first time" can't be obtained in the continuum, the moment before Y first appears.

Definition 16. If S is a dynamic set and $X, Y \subset \mathcal{P}_S$, $(X/Y) \equiv \{p \in \mathcal{P}_S : \text{for some } \bar{p}[\lambda_1, \lambda_2] \in S_{X/Y}, p = \bar{p}(\lambda_2)\}$

Theorem 17. If D is a homogeneous dynamic space then $D_{X/Y} = X \rightarrow (X/Y) \upharpoonright Y$

Proof. If $\bar{p}[\lambda_1, \lambda_2] \in D_{X/Y}$ then $\bar{p}(\lambda_1) \in X$, $\bar{p}(\lambda_2) \in (X/Y)$, and for all $\lambda \in [\lambda_1, \lambda_2)$, $\bar{p}(\lambda) \notin Y$, so $D_{X/Y} \subset X \rightarrow (X/Y) \upharpoonright Y$. (Note that this doesn't require D to be homogeneous, or a dynamic space.)

It remains to show that $X \rightarrow (X/Y) \upharpoonright Y \subset D_{X/Y}$. Take $\bar{p}[\lambda_1, \lambda_2] \in X \rightarrow (X/Y) \upharpoonright Y$, $p \equiv \bar{p}(\lambda_2) \in (X/Y)$. By the definition of (X/Y) , there's a $\bar{p}'[\lambda_3, \infty] \in p \rightarrow$ s.t. $\lambda_3 = \text{glb}(\bar{p}'[\lambda_3, \infty]^{-1}[Y])$. By homogeneity, there's a \bar{p}'' s.t. for all $\lambda \in \Lambda_D$, $\bar{p}''(\lambda) = \bar{p}'(\lambda + (\lambda_3 - \lambda_2))$ (assuming $\lambda_3 \geq \lambda_2$ or Λ_D is unbounded from below; otherwise $\bar{p}'(\lambda) = \bar{p}''(\lambda + (\lambda_2 - \lambda_3))$ may be asserted for all $\lambda \in \Lambda_D$). By considering $\bar{p}[\lambda_1, \lambda_2] \circ \bar{p}''[\lambda_2, \infty]$, it follows that $\bar{p}[\lambda_1, \lambda_2] \in D_{X/Y}$. \square

E. Limits and Closed Sets

In this final section, system dynamics will be used to define limits. Four ways in which a point $p \in \mathcal{P}_S$ can be a limit point of set $X \subset \mathcal{P}_S$ at λ will be defined. They correspond to: if the system is in state p at λ then X must be about to occur, if the system is in state p at λ then X might be about to occur, if the system is in state p at λ then X might have just happened, and if the system is in state p at λ then X must have just happened. For formally:

Definition 18. If S is a dynamic set, $p \in \mathcal{P}_S$, $\lambda \in \Lambda_S$, and X is a non-empty subset of \mathcal{P}_S then

- $p \in \lim_{\lambda}^{\triangleright} X$ if for every $\bar{p}[\lambda, \infty] \in p \rightarrow$, every $\lambda' > \lambda$, there's a $\lambda'' \in (\lambda, \lambda')$ s.t. $\bar{p}(\lambda'') \in X$
- $p \in \lim_{\lambda}^{>} X$ if for every $\bar{p}[\lambda, \infty] \in p \rightarrow$ and a $\lambda'' \in (\lambda, \lambda')$ s.t. $\bar{p}(\lambda'') \in X$
- $p \in \lim_{\lambda}^{<} X$ if $\lambda > \text{Min}(\Lambda_D)$ and for every $\bar{p}[-\infty, \lambda] \in \rightarrow p$ and a $\lambda'' \in (\lambda', \lambda)$ s.t. $\bar{p}(\lambda'') \in X$
- $p \in \lim_{\lambda}^{\triangleleft} X$ if $\lambda > \text{Min}(\Lambda_D)$ and for every $\bar{p}[-\infty, \lambda] \in \rightarrow p$ s.t. , every $\lambda' < \lambda$, there's a $\lambda'' \in (\lambda', \lambda)$ s.t. $\bar{p}(\lambda'') \in X$

If S is homogeneous, then the λ subscript may be dropped, and we may simply write $p \in \lim^{\triangleright} X$, etc.

Definition 19. If S is a dynamic set, $X \subset \mathcal{P}_S$, and $\lambda \in \Lambda_S$ then $X_{\lambda}^{\triangleright} \equiv X \cup \lim^{\triangleright}_{\lambda} X$, $X_{\lambda}^{>} \equiv X \cup \lim^{>}_{\lambda} X$, $X_{\lambda}^{<} \equiv X \cup \lim^{<}_{\lambda} X$, and $X_{\lambda}^{\triangleleft} \equiv X \cup \lim^{\triangleleft}_{\lambda} X$.

Once again, if S is homogeneous, then the λ subscript may be dropped, and one may simply write X^{\triangleright} , etc.

A set is closed if it contains all its limit points. The following theorem lays out conditions under which X^{\triangleright} , etc., are closed.

Theorem 20. 1) If S is a homogeneous dynamic set, and $X \subset \mathcal{P}_S$, then $X^{\triangleright\triangleright} = X^{\triangleright}$ and $X^{\triangleleft\triangleleft} = X^{\triangleleft}$.

1) If D is a homogeneous dynamic space, and $X \subset \mathcal{P}_D$, then $X^{>>} = X^>$ and $X^{<<} = X^<$.

Proof. If Λ_D is discrete, this holds because if Λ_D is discrete then for all $X \subset D$, $\lim^\square X = \emptyset$.

Take Λ_D to be continuous.

1) Assume $p \in \lim^\triangleright X^\triangleright$ and take $\bar{p}[\lambda, \infty] \in S_{p \rightarrow}$ & $\lambda' > \lambda$. There's a $\lambda'' \in (\lambda, \lambda + \frac{1}{2}(\lambda' - \lambda))$ s.t. $\bar{p}(\lambda'') \in X^\triangleright$. Because $\bar{p}(\lambda'') \in X^\triangleright$, either $\bar{p}(\lambda'') \in X$ or $\bar{p}(\lambda'') \in \lim^\triangleright X$, so there's a $\lambda''' \in [\lambda'', \lambda'' + \frac{1}{2}(\lambda' - \lambda))$ s.t. $\bar{p}(\lambda''') \in X$. Therefore there's a $\lambda''' \in (\lambda, \lambda')$ s.t. $\bar{p}(\lambda''') \in X$, and so $p \in \lim^\triangleright X$, and so $p \in X^\triangleright$.

\triangleleft is similar

2) Assume $p \in \lim^> X^>$. For any $\lambda, \lambda' > \lambda$ there's a $\bar{p}[\lambda, \infty] \in p \rightarrow$ and a $\lambda'' \in (\lambda, \lambda + \frac{1}{2}(\lambda' - \lambda))$ s.t. $\bar{p}(\lambda'') \in X^>$. Because $\bar{p}(\lambda'') \in X^>$, either $\bar{p}(\lambda'') \in X$ or $\bar{p}(\lambda'') \in \lim^> X$, so there's a $\bar{p}'[\lambda'', \infty] \in p \rightarrow$, $\lambda''' \in [\lambda'', \lambda'' + \frac{1}{2}(\lambda' - \lambda))$ s.t. $\bar{p}'(\lambda''') \in X$. Take $\bar{p}''[\lambda, \infty] = \bar{p}[\lambda, \lambda''] \circ \bar{p}'[\lambda'', \infty]$. $\bar{p}'' \in p \rightarrow$, $\lambda''' \in (\lambda, \lambda')$, and $\bar{p}''(\lambda''') \in X$, so $p \in \lim^\triangleright X$, and so $p \in X^\triangleright$.

$<$ is similar. □

Under the conditions of Theorem 20, the collection of all closed sets of type “ $>$ ” are closed under intersections and finite unions. They therefore form a topology on \mathcal{P}_D ; the lower-limit topology. The closed sets of type “ $<$ ” also form a topology; the upper-limit topology.

The closed sets of type “ \triangleright ”, and those of type “ \triangleleft ”, are closed under intersection, but not union. They therefore do not form topologies. That is because these limits are fairly strict, and so can not always determine whether or not some point p is local to some set X .

Theorem 21. If S is a homogeneous dynamic set, and $X, Y \subset \mathcal{P}_S$, then $(X/Y) \subset Y^>$

Proof. By the definition of (X/Y) , $p \in (X/Y)$ iff for some $\bar{p} \in S$, $\lambda_1, \lambda_2 \in \Lambda_D$ ($\lambda_1 \leq \lambda_2$), $\bar{p}(\lambda_2) = p$, $\bar{p}(\lambda_1) \in X$, $[\lambda_1, \lambda_2] \cap \bar{p}^{-1}[Y] = \emptyset$, and for all $\lambda' > \lambda_2$ $[\lambda_2, \lambda'] \cap \bar{p}^{-1}[Y] \neq \emptyset$. In this case, $\bar{p}[\lambda_2, \infty] \in p \rightarrow$ and for all $\lambda' > \lambda_2$ there's a $\lambda'' \in [\lambda_2, \lambda')$ s.t. $\bar{p}(\lambda'') \in Y$, so either $p \in Y$ or $p \in \lim^> Y$. □

III. EXPERIMENTS

A. Shells

In this part, experimentation on dynamic systems will be formalized. As mentioned in the Introduction, this will be analogous to creating a formal theory of computation via automata. In computation theory, sequences of characters are read into an automata, and the automata determines whether they are sentences in a given language. One of the goals of the theory is to determine what languages automata are capable of deciding. Experiments are similar; system paths are “read into” an experimental set-up, which then determines which outcomes these paths belong to. Our goal will be to determine what sets of outcomes experiments are capable of deciding.

In this first section, we will not divide the experiment into the system being experimented on, and the environment containing the experimental apparatus; for now, we consider the closed system that encompasses both. The dynamic space of this closed system will be used to formalize some of the external properties of experiments; namely, that experiments are re-runnable, they have a clearly defined start, once started they must complete, and once complete they “remember” that the experiment took place.

Definition 22. A *shell* is a triple, (D, I, F) , where D is a dynamic space and $I, F \subset \mathcal{P}_D$ are the sets of initial and final states. These must satisfy:

- 1) D is homogeneous
- 2) I is homogeneously realized
- 3) $I \rightarrow = (I \upharpoonright I \rightarrow F) \rightarrow$ (That is, $I \rightarrow = D_{I \upharpoonright I \rightarrow F} \rightarrow$)
- 4) $\rightarrow F = \rightarrow (I \upharpoonright I \rightarrow F)$

In the above definition, I represents the set of initial states, or start states; when the system enters into one of these states, the experiment starts. F represents the set of final states; when the system enters into one of these states, the experiment has ended.

Axioms (1) and (2) ensure that the experiment is reproducible. Axiom (3) ensures that once the experiment begins, it must end. Axiom (4) ensures that the system can only enter a final state via the experiment.

A note of explanation may be in order for this final axiom. D is assumed to be a closed system encompassing everything that has bearing on the experiment, including the person

performing the experiment. As a result, all recording equipment is considered part of D , including the experimenter's memory. So if the final axiom were violated, when the final state is reached, you wouldn't be able to remember whether or not the experiment had taken place.

In (3) and (4), $I \upharpoonright I \rightarrow F$ is used rather than $I \rightarrow F$ in order to ensure that there's a clearly defined space in which the experiment takes place. For example, $\rightarrow F = \rightarrow I \rightarrow F$ would allow for a path, $\bar{p}[-\infty, 0] \in \rightarrow F$, s.t. for all $n \in \mathbb{N}^+$, $\bar{p}(\frac{-1}{2n}) \in I$ and $\bar{p}(\frac{-1}{2n+1}) \in F$; in this case, the F state at $\lambda = 0$ can not be paired with any particular experimental run. Similarly, $I \rightarrow = I \rightarrow F \rightarrow$ would allow a path to cross I several times before crossing F ; in this case it would not be clear which crossing of I represented the start of the experiment.

An experiment is considered to be in progress while the space is in a path segment that runs from I to F . The set of states in those path segments constitute the shell interior, or more formally:

Definition 23. If $Z = (D, I, F)$ is a shell, $Int(Z) = \{p \in \mathcal{P}_D : I \rightarrow p \upharpoonright F \neq \emptyset\}$ is the *shell interior*.

Theorem 24. If $Z = (D, I, F)$ is a shell then $Int(Z) = Ran(D_{I/F}) = Ran(I \rightarrow (I/F) \upharpoonright F)$. (The abbreviation " $Ran(A)$ " stands for the set of states $\bigcup_{\bar{p}[x_1, x_2] \in A} Ran(\bar{p}[x_1, x_2])$.)

Proof. By Thm 17 and shell axiom 1, $D_{I/F} = I \rightarrow (I/F) \upharpoonright F$.

If $p \in Ran(I \rightarrow (I/F) \upharpoonright F)$ then for some $\bar{p}[\lambda_1, \lambda_2] \in I \rightarrow (I/F) \upharpoonright F$, some $\lambda_3 \in [\lambda_1, \lambda_2]$, $\bar{p}(\lambda_3) = p$, so $\bar{p}[\lambda_1, \lambda_3] \in I \rightarrow p \upharpoonright F$, and so $p \in Int(Z)$.

If $p \in Int(Z)$ then there's a $\bar{p}[\lambda_1, \lambda_2] \in I \rightarrow p \upharpoonright F$. Since $I \rightarrow = (I \upharpoonright I \rightarrow F) \rightarrow$ there must be a $\bar{p}'[\lambda_1, \lambda_3] \in D_{I/F}$ s.t. $\bar{p}'[\lambda_1, \lambda_2] = \bar{p}[\lambda_1, \lambda_2]$, so $p \in Ran(D_{I/F})$. \square

The following items will prove quite useful:

Definition 25. If $Z = (D, I, F)$ is a shell, $Dom(Z) \equiv F \bigcup Int(Z)$ is the *shell domain*

If $A \subset Dom(Z)$, $\omega_A \equiv \{\bar{p}[0, \lambda] \in I \upharpoonright I \rightarrow A\} = I \upharpoonright (0, I) \rightarrow A$

If $A \subset F$, $\Theta_A \equiv \{\bar{p}[0, \lambda] \in D_{I/F} : \text{for some } \bar{p}'[0, \lambda'] \in \omega_A, \bar{p}'[0, \lambda] = \bar{p}[0, \lambda]\}$

Given that our current state is in A , ω_A tells us what has happened in the experiment; because shell dynamics are homogeneous, and I is homogeneously realized, it is sufficient only consider paths that start at $\lambda_0 = 0$.

Given that the final state is in A , Θ_A captures what occurred during the experiment. The shell axioms validate these interpretations of ω_A and Θ_A .

B. E-Automata

Further structure will now be added to shells, starting with dividing the shell domain into a system and its environment.

1. Environmental Shells

In the following definition: “ \otimes ” will refer to the Cartesian product, and for n-ary relation, R , on $A_1 \otimes \dots \otimes A_n$, $P_i(R)$ refers the set of $a \in A_i$ s.t. for some $r \in R$, $a = r_i$.

Definition 26. An *environmental shell*, $Z = (D, I, F)$, is a shell together with three sets, \mathcal{S}_Z , $\mathcal{E}_{Int(Z)}$, and \mathcal{E}_F satisfying

- 1) $(\mathcal{S}_Z \otimes \mathcal{E}_{Int(Z)}) \cap \mathcal{P}_D = Int(Z)$
 - 2) $(\mathcal{S}_Z \otimes \mathcal{E}_F) \cap \mathcal{P}_D = F$
 - 3) $\mathcal{S}_Z = P_1(Dom(Z))$, $\mathcal{E}_{Int(Z)} = P_2(Int(Z))$, and $\mathcal{E}_F = P_2(F)$
- \mathcal{S}_Z is the *system* and $\mathcal{E}_Z \equiv \mathcal{E}_{Int(Z)} \cup \mathcal{E}_F$ is the *environment*.

Informally speaking, it’s assumed that the system (\mathcal{S}_Z) is being observed, measured, recorded, etc., and that any observers, measuring devices, recording equipment, etc., are in the environment. Therefore, it is components in the environment which decide whether the experiment is complete; this is why a subset of the environmental states, \mathcal{E}_F , determine whether the shell is on F .

If $\bar{p}[\lambda_1, \lambda_2]$ is a path segment from $I \upharpoonright I \rightarrow F$, $\bar{p}[\lambda_1, \lambda_2] \cdot \mathcal{S}$ is the system component of the path segment, and $\bar{p}[\lambda_1, \lambda_2] \cdot \mathcal{E}$ is the environmental portion. More formally:

Definition 27. For environmental shell Z :

If $\bar{s}[\lambda_1, \lambda_2] : [\lambda_1, \lambda_2] \rightarrow \mathcal{S}_Z$, $\bar{e}[\lambda_1, \lambda_2] : [\lambda_1, \lambda_2] \rightarrow \mathcal{E}_Z$, and $\bar{p}[\lambda_1, \lambda_2] = (\bar{s}[\lambda_1, \lambda_2], \bar{e}[\lambda_1, \lambda_2])$ then $\bar{p}[\lambda_1, \lambda_2] \cdot \mathcal{S} \equiv \bar{s}[\lambda_1, \lambda_2]$ and $\bar{p}[\lambda_1, \lambda_2] \cdot \mathcal{E} \equiv \bar{e}[\lambda_1, \lambda_2]$.

If A is any set of shell domain path segments, $A \cdot \mathcal{S} \equiv \{\bar{p}[\lambda_1, \lambda_2] \cdot \mathcal{S} : \bar{p}[\lambda_1, \lambda_2] \in A\}$ and $A \cdot \mathcal{E} \equiv \{\bar{p}[\lambda_1, \lambda_2] \cdot \mathcal{E} : \bar{p}[\lambda_1, \lambda_2] \in A\}$.

This can be applied to ω_A and Θ_A to extract the system information:

Definition 28. If Z is an environmental shell:

$$\text{If } X \subset \mathcal{E}_F, \mathcal{O}_X \equiv \Theta_{\mathcal{S}_Z} \otimes_X \cdot \mathcal{S}$$

If $A \subset \text{Int}(Z)$:

$$\Sigma_A \equiv \{\bar{s}[0, \lambda_1] : \text{for some } \bar{p}[0, \lambda_2] \in \Theta_F, \bar{p}(\lambda_1) \in A \text{ \& } \bar{p}[0, \lambda_1] \cdot \mathcal{S} = \bar{s}[0, \lambda_1]\}$$

$$\Sigma_{A \rightarrow} \equiv \{\bar{s}[\lambda_1, \lambda_2] : \text{for some } \bar{p}[0, \lambda_2] \in \Theta_F, \bar{p}(\lambda_1) \in A \text{ \& } \bar{p}[\lambda_1, \lambda_2] \cdot \mathcal{S} = \bar{s}[\lambda_1, \lambda_2]\}.$$

If an experiment completes with the environment in X , \mathcal{O}_X tells us what happened in the system during the experimental run (the “ \mathcal{O} ” stands for “outcome”). $\mathcal{O}_{\mathcal{E}_F}$ may be abbreviated \mathcal{O}_F .

Σ_A and $\Sigma_{A \rightarrow}$ give insight into what’s happening to the system in the shell interior: Σ_A gives the system paths from I to A and $\Sigma_{A \rightarrow}$ gives the system paths from A to (I/F) . For convenience, $\Sigma_\sigma \otimes_{\mathcal{E}_{\text{Int}(Z)}} \cdot$ may be written Σ_σ .

Theorem 29. $\mathcal{O}_F = \Sigma_{(I/F)}$

Proof. It’s clear that $\mathcal{O}_F \subset \Sigma_{(I/F)}$.

Assume $\bar{s}[0, \lambda] \in \Sigma_{(I/F)}$. For some $\bar{p}[0, \lambda_2] \in \Theta_F$, $\bar{p}(\lambda_1) \equiv p \in (I/F)$ and $\bar{p}[0, \lambda_1] \cdot \mathcal{S} = \bar{s}[0, \lambda_1]$. Since $p \in (I/F)$ and D is homogeneous there must be a $\bar{p}'[\lambda_1, \infty] \in p \rightarrow$ s.t. for all $\lambda_2 > \lambda_1$ there’s a $\lambda' \in [\lambda_1, \lambda_2]$ s.t. $\bar{p}'(\lambda') \in F$. Therefore $\bar{p}[0, \lambda_1] \circ \bar{p}'[\lambda_1, \infty] \in I \rightarrow$ and $\bar{p}[0, \lambda_1] \in D_{I/F}$; this means that $\bar{p}[0, \lambda_1] \in \Theta_F$, and so $\bar{s}[0, \lambda_1] \in \mathcal{O}_F$. \square

2. The Automata Condition

The automata condition demands that an experiment will terminate based solely on what has transpired in the system. In particular, it specifies that if $\bar{p}_1[0, \lambda_1], \bar{p}_2[0, \lambda_2] \in \omega_F$, $\bar{p}_1[0, \lambda] \in \Theta_F$, and $\bar{p}_1[0, \lambda] \cdot \mathcal{S} = \bar{p}_2[0, \lambda] \cdot \mathcal{S}$ then $\bar{p}_2[0, \lambda] \in \Theta_F$. Given shell axiom (3), this may be rephrased as follows:

Definition 30. An environmental shell satisfies *the automata condition* if for every $\bar{s}_1[0, \lambda_1], \bar{s}_2[0, \lambda_2] \in \mathcal{O}_F$ s.t. $\lambda_2 > \lambda_1$, $\bar{s}_1[0, \lambda_1] \neq \bar{s}_2[0, \lambda_1]$

The constraints this places on the closed system’s dynamics are summarized by the following theorem.

Theorem 31. *The following assertions on environmental shell Z are equivalent:*

- 1) Z satisfies the automata condition

2) For $p \in \text{Int}(Z)$, if $\Sigma_p \cap \mathcal{O}_F \neq \emptyset$ then $p \in F^\triangleright$

3) $(I/F) \subset F^\triangleright$ and $\Sigma_{\text{Int}(Z) - (I/F)} \cap \Sigma_{(I/F)} = \emptyset$

Proof. $1 \Rightarrow 2$: In order for the automata condition to hold, If $p \in \text{Int}(Z)$ and $\Sigma_p \cap \mathcal{O}_F \neq \emptyset$ then either $p \in F \cap (I/F)$, or all paths leaving p must immediately enter F ; either way, $p \in F^\triangleright$.

$2 \Rightarrow 3$: Since $\mathcal{O}_F = \Sigma_{(I/F)}$, $\Sigma_{(I/F)} \cap \mathcal{O}_F \neq \emptyset$, and so $(I/F) \subset F^\triangleright$.

If $p \in \text{Int}(Z) - (I/F)$ then the shell is not in F and is not about to transition into F , so $p \notin F^\triangleright$, and so $\Sigma_p \cap \mathcal{O}_F = \emptyset$.

$3 \Rightarrow 1$: Take $\bar{p}_1[0, \lambda_1], \bar{p}_2[0, \lambda_2] \in \Theta_F$ and $\lambda_2 \geq \lambda_1$. If $\bar{p}_1[0, \lambda_1] \cdot \mathcal{S} = \bar{p}_2[0, \lambda_1] \cdot \mathcal{S}$ then $\Sigma_{\bar{p}_1(\lambda_1)} \cap \Sigma_{\bar{p}_2(\lambda_1)} \neq \emptyset$ and so $\bar{p}_2(\lambda_1) \in (I/F)$ (since $\bar{p}_1(\lambda_1) \in (I/F)$ and $\Sigma_{\text{Int}(Z) - (I/F)} \cap \Sigma_{(I/F)} = \emptyset$). Since $\bar{p}_2(\lambda_1) \in (I/F)$, $\bar{p}_2(\lambda_1) \in F^\triangleright$, and so $\bar{p}_2[0, \lambda_1] \in \Theta_F$, and for all $\lambda_3 > \lambda_1$ $\bar{p}_2[0, \lambda_3] \notin \Theta_F$. Therefore $\lambda_2 = \lambda_1$. So if $\bar{p}_1[0, \lambda_1], \bar{p}_2[0, \lambda_2] \in \Theta_F$ and $\lambda_2 > \lambda_1$ then $\bar{p}_1[0, \lambda_1] \cdot \mathcal{S} \neq \bar{p}_2[0, \lambda_1] \cdot \mathcal{S}$ \square

3. Unbiased Conditions

An environmental shell is unbiased if the environment does not influence the outcome. In the strongest sense this demands that, while the environment may record the system's past, it has no effect on the system's future. Naively, this may be written: For every $(s, e_1), (s, e_2) \in \text{Int}(Z)$, $\Sigma_{(s, e_1) \rightarrow} = \Sigma_{(s, e_2) \rightarrow}$. However, because in general $\Sigma_{(s, e_1)} \neq \Sigma_{(s, e_2)}$, the elements of $\Sigma_{(s, e_1) \rightarrow}$ and $\Sigma_{(s, e_2) \rightarrow}$ may terminate at different times; this leads to a definition whose wording is more convoluted, but whose meaning is essentially the same.

Definition 32. An environmental shell, (D, I, F) is *strongly unbiased* if for every $\lambda \in \text{Dom}(\Theta_F)$, every $(s, e_1), (s, e_2) \in \Theta_F(\lambda)$, $\bar{s}_1[\lambda, \lambda_1] \in \Sigma_{(s, e_1) \rightarrow}$ iff there exists a $\bar{s}_2[\lambda, \lambda_2] \in \Sigma_{(s, e_2) \rightarrow}$ s.t., with $\lambda' \equiv \text{Min}(\lambda_1, \lambda_2)$, $\bar{s}_1[\lambda, \lambda'] = \bar{s}_2[\lambda, \lambda']$.

This condition means that any effect the environment may have on the system can be incorporated into the system dynamics, allowing the system to be comprehensible without having to reference its environment. The following theorem shows that if an environmental shell is strongly unbiased, \mathcal{O}_F behaves like a dynamic space.

Theorem 33. *If an environmental shell is strongly unbiased then for every $\bar{s}_1[0, \lambda_1], \bar{s}_2[0, \lambda_2] \in \mathcal{O}_F$, s.t. for some $\lambda \in [0, \text{Min}(\lambda_1, \lambda_2)]$, $\bar{s}_1(\lambda) = \bar{s}_2(\lambda)$, there exists an $\bar{s}_3[0, \lambda_3] \in \mathcal{O}_F$ s.t., with $\lambda' \equiv \text{Min}(\lambda_2, \lambda_3)$, $\bar{s}_3[0, \lambda'] = \bar{s}_1[0, \lambda] \circ \bar{s}_2[\lambda, \lambda']$*

Proof. First note that, regardless of whether the environmental shell is unbiased, for any $p \in \text{Int}(Z)$, if $\bar{s}[0, \lambda] \in \Sigma_p$ and $\bar{s}'[\lambda, \lambda'] \in \Sigma_{p \rightarrow}$ then $\bar{s}[0, \lambda] \circ \bar{s}'[\lambda, \lambda'] \in \mathcal{O}_F$.

Taking $s = \bar{s}_1(\lambda)$, for some $e_1, e_2 \in \mathcal{E}_{\text{Int}(Z)}$, $\bar{s}_1[0, \lambda] \in \Sigma_{(s, e_1)}$ and $\bar{s}_2[\lambda, \lambda_2] \in \Sigma_{(s, e_2) \rightarrow}$. Since the shell is unbiased, there is a $\bar{s}_3[\lambda, \lambda_3] \in \Sigma_{(s, e_1) \rightarrow}$ s.t. $\bar{s}_3[\lambda, \lambda'] = \bar{s}_2[\lambda, \lambda']$ ($\lambda' \equiv \text{Min}(\lambda_2, \lambda_3)$). By the considerations of the prior paragraph, $\bar{s}_1[0, \lambda] \circ \bar{s}_3[\lambda, \lambda_3] \in \mathcal{O}_F$. \square

There are experiments for which the strong unbiased condition can fail, but the experiment still be accepted as valid. Taking an example from quantum mechanics, consider the case in which a particle's position is measured λ_1 , and if the particle is in region A , the particle's spin will be measured along the y-axis at λ_2 , and if particle is not in region A , the spin will be measured along the z-axis at λ_2 . Now consider two paths, \bar{s}_1 and \bar{s}_2 , s.t. \bar{s}_1 is in region A at λ_1 , \bar{s}_2 is not in region A at λ_1 , and for some $\lambda \in (\lambda_1, \lambda_2)$, $\bar{s}_1(\lambda) = \bar{s}_2(\lambda) \equiv s$. $\bar{s}_1[-\infty, \lambda] \circ \bar{s}_2[\lambda, \infty]$ is not a possible path because a particle can't be in region A at λ_1 and have its spin polarized along the z-axis at λ_2 . Therefore the set of particle paths is not a dynamic space (though it is still a dynamic set). Since the strong unbiased condition insures that the system can be described by a dynamic space, the unbiased condition must have failed.

To see this, assume that the e-automata is in state (s, e_1) at λ in the case where the system takes path \bar{s}_1 , and state (s, e_2) in the case where the system takes path \bar{s}_2 . e_1 and e_2 determine different futures for the particle paths, e_1 insures that the spin will be measured along the y-axis at λ_2 while e_2 insures that the spin will be measured along the z-axis at λ_2 . This violates the strong unbiased condition. However, it doesn't necessarily create a problem for the experiment because e_1 and e_2 know enough about the system history to know $s_1[-\infty, \lambda]$ and $s_2[-\infty, \lambda]$ must reside in separate outcomes (one belongs to the set of "A" outcomes, the other to the set of "not A" outcomes); once paths have been differentiated into separate outcomes, they may be treated differently by the environment. This motivates a weak version of the unbiased condition, one that holds only when (s, e_1) and (s, e_2) have not yet been differentiated into separate outcomes.

For an environmental shell, $Z = (D, I, F)$, $p, p' \in \text{Int}(Z)$ have not been differentiated

if for some $e \in \mathcal{E}_F$, $\lambda \in \Lambda$, $\Sigma_p \cap \mathcal{O}_e[0, \lambda] \neq \emptyset$ and $\Sigma_{p'} \cap \mathcal{O}_e[0, \lambda] \neq \emptyset$. If (s, e) and (s, e') have not been differentiated then the unbiased condition should hold for them. Similarly, if (s, e_1) and (s, e_2) have not been differentiated, and neither have (s, e_2) and (s, e_3) , then the unbiased condition should hold for (s, e_1) and (s, e_3) . This motivates the following definition

Definition 34. For e-automata Z , $e \in \mathcal{E}_F$, $\lambda \geq 0$

$$|\mathcal{O}_e[0, \lambda]|_0 \equiv \mathcal{O}_e[0, \lambda]$$

$$|\mathcal{O}_e[0, \lambda]|_{n+1} \equiv \{\bar{s}[0, \lambda] : \text{for some } e' \in \mathcal{E}_F, \mathcal{O}_{e'}[0, \lambda] \cap |\mathcal{O}_e[0, \lambda]|_n \neq \emptyset, \& \bar{s}[0, \lambda] \in \mathcal{O}_{e'}[0, \lambda]\}$$

$$|\mathcal{O}_e[0, \lambda]| \equiv \bigcup_{n \in \mathbb{N}} |\mathcal{O}_e[0, \lambda]|_n$$

Note that if $\bar{s}[0, \lambda_0] \in \mathcal{O}_e$ then $\bar{s}[0, \lambda] \in \mathcal{O}_e[0, \lambda]$ iff $\lambda_0 \geq \lambda$.

Definition 35. For environmental shell, $Z = (D, I, F)$, $p, p' \in \text{Int}(Z)$ are *undifferentiated* if for some $e \in \mathcal{E}_F$, $\lambda \in \Lambda$, $\Sigma_p \cap |\mathcal{O}_e[0, \lambda]| \neq \emptyset$ and $\Sigma_{p'} \cap |\mathcal{O}_e[0, \lambda]| \neq \emptyset$

Now define the weak unbiased condition in the same way as the strong condition, except that it only applies to undifferentiated states.

Definition 36. An environmental shell, (D, I, F) is *weakly unbiased* if for every $\lambda \in \text{Dom}(\Theta_F)$, every $(s, e_1), (s, e_2) \in \Theta_F(\lambda)$ s.t. (s, e_1) and (s, e_2) are undifferentiated, $\bar{s}_1[\lambda, \lambda_1] \in \Sigma_{(s, e_1) \rightarrow}$ iff there exists a $\bar{s}_2[\lambda, \lambda_2] \in \Sigma_{(s, e_2) \rightarrow}$ s.t. with $\lambda' \equiv \text{Min}(\lambda_1, \lambda_2)$, $\bar{s}_1[\lambda, \lambda'] = \bar{s}_2[\lambda, \lambda']$.

Theorem 37. *If an environmental shell is weakly unbiased then for every $\bar{s}_1[0, \lambda_1], \bar{s}_2[0, \lambda_2] \in \mathcal{O}_F$, s.t. for some $\lambda \in [0, \text{Min}(\lambda_1, \lambda_2)]$, $\bar{s}_1(\lambda) = \bar{s}_2(\lambda) = s$ and for some $e \in \mathcal{E}_F$, $\bar{s}_1[0, \lambda], \bar{s}_2[0, \lambda] \in |\mathcal{O}_e[0, \lambda]|$, there exists an $\bar{s}_3[0, \lambda_3] \in \mathcal{O}_F$ s.t., with $\lambda' \equiv \text{Min}(\lambda_2, \lambda_3)$, $\bar{s}_3[0, \lambda'] = \bar{s}_1[0, \lambda] \circ \bar{s}_2[\lambda, \lambda']$*

Proof. Choose $e_1, e_2 \in \mathcal{E}_{\text{Int}(Z)}$ s.t. $\bar{s}_1[0, \lambda] \in \Sigma_{(s, e_1)}$, $\bar{s}_2[\lambda, \lambda_2] \in \Sigma_{(s, e_2) \rightarrow}$, $\Sigma_{(s, e_1)} \cap |\mathcal{O}_e[0, \lambda]| \neq \emptyset$, and $\Sigma_{(s, e_2)} \cap |\mathcal{O}_e[0, \lambda]| \neq \emptyset$. Such e_1 and e_2 must exist because $\bar{s}_1[0, \lambda], \bar{s}_2[0, \lambda] \in |\mathcal{O}_e[0, \lambda]|$. (s, e_1) and (s, e_2) are undifferentiated, and so the proof now proceeds similarly to that for Thm 33. \square

4. E-Automata

Definition 38. An environmental shell is an *e-automata* if it is weakly unbiased and satisfies the automata condition.

Theorem 39. *If Z is an e-automata, $p_1, p_2 \in \text{Int}(Z)$, and $\Sigma_{p_1} \cap \Sigma_{p_2} \neq \emptyset$ then $\Sigma_{p_1 \rightarrow} = \Sigma_{p_2 \rightarrow}$*

Proof. Since $\Sigma_{p_1} \cap \Sigma_{p_2} \neq \emptyset$, p and p' are undifferentiated, so the result follows from Z being weakly unbiased & satisfying the automata condition \square

As mentioned earlier, an important goal will be to determine what sets of outcomes can be decided by various types of e-automata. Towards that end, the meaning of an e-automata “deciding a set of outcomes” will now be defined. For the moment we’ll concentrate measurements on dynamic spaces; the more general case will be taken up in a later section.

Definition 40. If X is a set, C is a *covering* of X if it is a set of subsets of X and $\bigcup C = X$. (Note: it is more common to demand that $X \subset \bigcup C$; the restriction to $\bigcup C = X$ will allow for some mild simplification.)

C is a *partition* of X if it is a pairwise disjoint covering

...

Definition 41. If $Z = (D, I, F)$ is an e-automata, $\lambda \in \Lambda_D$, $A \subset F$, and $X \subset \mathcal{E}_F$:

$$\Theta_A^\lambda \equiv \{\bar{p}[\lambda, \lambda'] \in D_{I/F} : \text{for some } \bar{p}'[\lambda, \lambda''] \in I \upharpoonright I \rightarrow A, \bar{p}'[\lambda, \lambda'] = \bar{p}[\lambda, \lambda']\}$$

$$\mathcal{O}_X^\lambda \equiv \Theta_{S_Z \otimes X}^\lambda \cdot \mathcal{S}$$

Because D is homogeneous and I is homogeneously realized, Θ_A^λ and \mathcal{O}_X^λ are simply Θ_A and \mathcal{O}_X shifted by λ , and theorems for Θ_A and \mathcal{O}_X can be readily translated to theorems for Θ_A^λ and \mathcal{O}_X^λ .

When an e-automata decides a set of outcomes on a dynamic space, it is fairly straightforward to define the relationship between the dynamic space and the e-automata’s outcomes.

Definition 42. A covering, K , of dynamic space D is *decided* by e-automata $Z = (D_Z, I, F)$ if $\Lambda_D = \Lambda_{D_Z}$ and, for some $\lambda \in \Lambda_D$, $K \equiv \{\rightarrow \mathcal{O}_e^\lambda \rightarrow : e \in \mathcal{E}_F\}$.

Note that Z can not decide K if starts after some λ . This may seem to violate experimental reproducibility. In a later section it will be seen that any such a measurement can be consistent with experimental reproducibility. It’s not difficult to see how. If we were interested, for example, in the likelihood that a system is in state y at time 1 *sec* given that it was in state x at time 0, we do not mean that we are only interested in those probabilities at a particular moment in the history of the universe; it is assumed that the system can be

assembled at any time, and the time at which it is assembled is simply assigned the value of 0.

If D is a dynamic space, and $A \subset D$, an e-automata can determine whether or not A occurs if it decides a covering on D , K , s.t. and for all $\alpha \in K$, either $\alpha \subset A$ or $\alpha \cap A = \emptyset$. While this concept of measurement is adequate for many purposes, it's a little too broad for quantum physics; in the next section a subclass of e-automata will be introduced that will eliminate the problematic measurements.

C. Ideal E-Automata

As they stand, e-automata draw no clear distinction between two different types of uncertainty: extrinsic uncertainty - uncertainty about the state of the environment, particularly with regard to knowing precisely which $e \in \mathcal{E}_F$ occurred, and intrinsic uncertainty - uncertainty about the system given complete knowledge of the environment.

It is often necessary to treat with these two type of uncertainty separately. Such is the case in quantum physics: if \bar{s}_1 and \bar{s}_2 are system paths, there's a difference between "either \bar{s}_1 or \bar{s}_2 was measured, but we don't know which" (extrinsic uncertainty) and " $\{\bar{s}_1, \bar{s}_2\}$ was measured" (intrinsic uncertainty), because in general $P(A \cup B) \neq P(A) + P(B)$.

In this section two properties will be introduced that together will remove the ambiguity between these types of uncertainty. These are the only refinement to e-automata that will be required.

1. Boolean E-Automata

Definition 43. An e-automata is *boolean* if for every $e, e' \in \mathcal{E}_F$ either $\mathcal{O}_e = \mathcal{O}_{e'}$ or $\mathcal{O}_e \cap \mathcal{O}_{e'} = \emptyset$.

Theorem 44. *If K is decided by a boolean e-automata then it is a partition*

Proof. Follows from the boolean & automata conditions. □

If an e-automata is not boolean, it's possible to have $e, e' \in \mathcal{E}_F$ s.t. $\mathcal{O}_e \neq \mathcal{O}_{e'}$ and $\mathcal{O}_e \cap \mathcal{O}_{e'} \neq \emptyset$. If it's known that e occurred, and so e' did not occur, it's not clear whether the elements of $\mathcal{O}_e \cap \mathcal{O}_{e'}$ could have occurred. Boolean e-automata eliminate this sort of

ambiguity; if e occurred and $\mathcal{O}_e \neq \mathcal{O}_{e'}$ then none of the paths in $\mathcal{O}_{e'}$ could have been taken. This means that for boolean e-automata, one can always separate uncertainty about which path(s) corresponding to \mathcal{O}_e occurred from uncertainty about which $e \in \mathcal{E}_F$ occurred.

Definition 45. For a boolean e-automata, $e \in \mathcal{E}_F$, $[e] \equiv \{e' \in \mathcal{E}_F : \mathcal{O}_e = \mathcal{O}_{e'}\}$

2. All-Reet E-Automata

It is possible for the environment to record a piece of information about the system at one point, and then later forget it, so that it is not ultimately reflected in the experimental outcome. This leads ambiguity of the type discussed above; should the outcome be understood to be as $A \cup B$ or as $A \text{ or } B$ (or as something distinct from either)?

An all-reet e-automata is all retaining, once it records a bit of information about the system, it never forgets it. In order to define this type of e-automate, it will be helpful to first define the subsets of $\omega_{Dom(Z)}$, $\Sigma_{Int(Z)}$, Θ_F , and \mathcal{O}_F containing just those path-segments that are consistent with a given environmental path-segment.

Definition 46. For $\bar{e} : \Lambda \rightarrow \mathcal{E}_Z$,

$$\omega_{\bar{e}[0, \lambda]} \equiv \{\bar{p}[0, \lambda] \in \omega_{Dom(Z)} : \bar{p}[0, \lambda] \cdot \mathcal{E} = \bar{e}[0, \lambda]\}$$

$$\Sigma_{\bar{e}[0, \lambda]} \equiv \{\bar{s}[0, \lambda] : (\bar{s}[0, \lambda], \bar{e}[0, \lambda]) \in \omega_{Int(Z)}\}$$

$$\text{If } \bar{e}(\lambda) \in \mathcal{E}_F:$$

$$\Theta_{\bar{e}[0, \lambda]} \equiv \{\bar{p}[0, \lambda'] \in \Theta_F : \text{for some } \bar{p}'[0, \lambda] \in \omega_{\bar{e}[0, \lambda]}, \bar{p}'[0, \lambda'] = \bar{p}[0, \lambda']\}$$

$$\mathcal{O}_{\bar{e}[0, \lambda]} \equiv \Theta_{\bar{e}[0, \lambda]} \cdot \mathcal{S}$$

Theorem 47. If Z is an e-automata and $e \in \mathcal{E}_F$ then $\mathcal{O}_e = \bigcup_{\bar{e}[0, \lambda] \in \omega_e \cdot \mathcal{E}} \mathcal{O}_{\bar{e}[0, \lambda]}$

Proof. Since $F = (\mathcal{S}_Z \otimes \mathcal{E}_F) \cap \mathcal{P}_D$, for any $(\bar{s}_1[0, \lambda], \bar{e}[0, \lambda]), (\bar{s}_2[0, \lambda], \bar{e}[0, \lambda]) \in \omega_F$, any $\lambda' \leq \lambda$, $\bar{s}_1[0, \lambda'] \in \mathcal{O}_{\bar{e}(\lambda)}$ iff $\bar{s}_2[0, \lambda'] \in \mathcal{O}_{\bar{e}(\lambda)}$; therefore $\bigcup_{\bar{e}[0, \lambda] \in \omega_e \cdot \mathcal{E}} \mathcal{O}_{\bar{e}[0, \lambda]} \subset \mathcal{O}_e$.

If $\bar{s}[0, \lambda'] \in \mathcal{O}_e$ then there must exist a $\bar{p}[0, \lambda] \in \omega_e$ s.t. consistent $\bar{s}[0, \lambda'] = \bar{p}[0, \lambda'] \cdot \mathcal{S}$, in which case $\bar{s}[0, \lambda'] = \mathcal{O}_{\bar{p}[0, \lambda'] \cdot \mathcal{E}}$. Therefore $\mathcal{O}_e \subset \bigcup_{\bar{e}[0, \lambda] \in \omega_e \cdot \mathcal{E}} \mathcal{O}_{\bar{e}[0, \lambda]}$. \square

Let's assume that as an experiment unfolds, the environment sequentially writes everything it discovers about the system to incorruptible memory. The state of the environment would include the state of this memory, so for every $\bar{e}_1[0, \lambda_1], \bar{e}_2[0, \lambda_2] \in \omega_F \cdot \mathcal{E}$, if $\mathcal{O}_{\bar{e}_1[0, \lambda_1]} \neq \mathcal{O}_{\bar{e}_2[0, \lambda_2]}$ then $\bar{e}_1(\lambda_1) \neq \bar{e}_2(\lambda_2)$. Since $\mathcal{O}_e = \bigcup_{\bar{e}[0, \lambda] \in \omega_e \cdot \mathcal{E}} \mathcal{O}_{\bar{e}[0, \lambda]}$, we are lead to the following definition.

Definition 48. For e-automata $Z = (D, I, F)$, $e \in \mathcal{E}_F$ is *reet* if for all $\bar{e}[0, \lambda] \in \omega_e \cdot \mathcal{E}$, $\mathcal{O}_{\bar{e}[0, \lambda]} = \mathcal{O}_e$

Z is *all-reet* if every $e \in \mathcal{E}_F$ is reet.

Theorem 49. If Z is an all-reet e-automata, $e \in \mathcal{E}_F$, $\bar{e}[0, \lambda] \in \omega_e \cdot \mathcal{E}$, and $\bar{e}[0, \lambda](\lambda') = e' \in \mathcal{E}_F$ then $\mathcal{O}_e = \mathcal{O}_{e'}$

Proof. Take $\lambda_1 = \text{glb}(\bar{e}^{-1}[0, \lambda][\mathcal{E}_F])$; since $\bar{e}[0, \lambda](\lambda') \in \mathcal{E}_F$, $\lambda_1 \leq \lambda'$. $\Theta_{\bar{e}[0, \lambda]} = \Theta_{\bar{e}[0, \lambda']} = \omega_{\bar{e}[0, \lambda_1]}$, and so $\mathcal{O}_e = \mathcal{O}_{\bar{e}[0, \lambda]} = \mathcal{O}_{\bar{e}[0, \lambda']} = \mathcal{O}_{e'}$. \square

Theorem 50. If Z is an all-reet e-automata then for every $e \in \mathcal{E}_F$, every $\bar{s}_1[0, \lambda_1], \bar{s}_2[0, \lambda_2] \in \mathcal{O}_e$, $\lambda_1 = \lambda_2$

Proof. Take any $\bar{e}[0, \lambda] \in \omega_e \cdot \mathcal{E}$; $\mathcal{O}_e = \mathcal{O}_{\bar{e}[0, \lambda]}$. Assume that $\lambda_1 < \lambda_2$; since $(I/F) \subset F^\triangleright$ there's a $\lambda' \in [\lambda_1, \lambda_2]$ s.t. $\bar{e}(\lambda') \in \mathcal{E}_F$. However, in that case $\bar{s}_2[0, \lambda_2] \in \mathcal{O}_e$ and $\bar{s}_2[0, \lambda_2] \notin \mathcal{O}_{\bar{e}[0, \lambda']} = \mathcal{O}_{\bar{e}(\lambda')}$, contrary to Thm 49. \square

Definition 51. If Z is an all-reet e-automata, $e \in \mathcal{E}_F$ and $\bar{s}[0, \lambda] \in \mathcal{O}_e$ then $\Lambda(\mathcal{O}_e) \equiv \lambda$.

Thm 50 implies a similar statement for all of $\text{Int}(Z)$.

Theorem 52. If $Z = (D, I, F)$ is an all-reet e-automata then for every $p \in \text{Int}(Z)$, every $\bar{s}_1[0, \lambda_1], \bar{s}_2[0, \lambda_2] \in \Sigma_p$, $\lambda_1 = \lambda_2$

Proof. Assume $\lambda_1 \leq \lambda_2$. Take any $\bar{p}_1[0, \lambda_1], \bar{p}_2[0, \lambda_2] \in \omega_p$ and any $\bar{p}[\lambda_1, \lambda_3] \in I \upharpoonright p \rightarrow F$. Because D is homogeneous there must be a $\bar{p}'[\lambda_2, \lambda_3 + \lambda_2 - \lambda_1] \in I \upharpoonright p \rightarrow F$ s.t. for all $\lambda \in [\lambda_1, \lambda_3]$, $\bar{p}(\lambda) = \bar{p}'(\lambda + \lambda_2 - \lambda_1)$. Define $\bar{p}_3[0, \lambda_3] \equiv \bar{p}_1[0, \lambda_1] \circ \bar{p}[\lambda_1, \lambda_3]$ and $\bar{p}_4[0, \lambda_3 + \lambda_2 - \lambda_1] \equiv \bar{p}_2[0, \lambda_2] \circ \bar{p}'[\lambda_2, \lambda_3 + \lambda_2 - \lambda_1]$. If $\bar{p}_3[0, \lambda_x] \in \Theta_F$ then $\bar{p}_4[0, \lambda_x + \lambda_2 - \lambda_1] \in \Theta_F$, so by Thm 50, $\lambda_1 = \lambda_2$. consistent \square

The next two theorems will prove to be quite useful.

Theorem 53. If Z is an all-reet e-automata then for any $e \in \mathcal{E}_F$, $\lambda \in [0, \Lambda(\mathcal{O}_e)]$, $\mathcal{O}_e = \mathcal{O}_e[0, \lambda] \circ \mathcal{O}_e[\lambda, \Lambda(\mathcal{O}_e)]$.

Proof. Clearly $\mathcal{O}_e \subset \mathcal{O}_e[0, \lambda] \circ \mathcal{O}_e[\lambda, \Lambda(\mathcal{O}_e)]$.

Take any $\bar{s}[0, \Lambda(\mathcal{O}_e)], \bar{s}'[0, \Lambda(\mathcal{O}_e)] \in \mathcal{O}_e$ s.t. $\bar{s}(\lambda) = \bar{s}'(\lambda)$. For $\bar{e}[0, \lambda'] \in \omega_e \cdot \mathcal{E}$, $\mathcal{O}_e = \mathcal{O}_{\bar{e}[0, \lambda']} = \omega_{\bar{e}[0, \Lambda(\mathcal{O}_e)]} \cdot \mathcal{S}$. $(\bar{s}[0, \lambda], \bar{e}[0, \lambda]) \circ (\bar{s}_1[\lambda, \Lambda(\mathcal{O}_e)], \bar{e}[\lambda, \Lambda(\mathcal{O}_e)]) \in \omega_{\bar{e}[0, \Lambda(\mathcal{O}_e)]}$, so since $\mathcal{O}_e = \omega_{\bar{e}[0, \Lambda(\mathcal{O}_e)]} \cdot \mathcal{S}$, $\bar{s}[0, \lambda] \circ \bar{s}[\lambda, \Lambda(\mathcal{O}_e)] \in \mathcal{O}_e$. \square

Theorem 54. *If Z is an all-reet e -automata then for every $e \in \mathcal{E}_F$, every $\lambda \in [0, \Lambda(\mathcal{O}_e)]$, every $e_\lambda \in (\Theta_e \cdot \mathcal{E})(\lambda)$, $\mathcal{O}_e[0, \lambda] = \bigcup_{s \in \mathcal{O}_e(\lambda)} \Sigma_{(s, e_\lambda)}$*

Proof. First note that there exists a $\bar{e}[0, \lambda'] \in \omega_e \cdot \mathcal{E}$ s.t. $\bar{e}(\lambda) = e_\lambda$.

A: $\bigcup_{s \in \mathcal{O}_e(\lambda)} \Sigma_{(s, e_\lambda)} \subset \mathcal{O}_e[0, \lambda]$

- Take any $s_\lambda \in \mathcal{O}_e(\lambda)$. Since Z is all-reet, $s_\lambda \in \mathcal{O}_{\bar{e}[0, \lambda']}(\lambda)$, so there must be a $\bar{p}'[0, \lambda'] \in \omega_{\bar{e}[0, \lambda']}$ s.t. $\bar{p}'(\lambda) = (s_\lambda, e_\lambda)$. Since $\bar{p}'[0, \lambda'] \in \omega_{\bar{e}[0, \lambda']}$, $\bar{p}[\lambda, \lambda'] \in I \upharpoonright (s_\lambda, e_\lambda) \rightarrow e$. Now take any $\bar{s}[0, \lambda] \in \Sigma_{(s_\lambda, e_\lambda)}$; there must exist a $(\bar{s}[0, \lambda], \bar{e}'[0, \lambda]) \in \omega_{(s_\lambda, e_\lambda)}$. $(\bar{s}[0, \lambda], \bar{e}'[0, \lambda]) \circ \bar{p}[\lambda, \Lambda(\mathcal{O}_e)] \in \Theta_e$ so $\bar{s}[0, \lambda] \in \mathcal{O}_e[0, \lambda]$. -

B: $\mathcal{O}_e[0, \lambda] \subset \bigcup_{s \in \mathcal{O}_e(\lambda)} \Sigma_{(s, e_\lambda)}$

- If $\bar{s}[0, \lambda] \in \mathcal{O}_e[0, \lambda]$ then $\bar{s}[0, \lambda] \in \mathcal{O}_{\bar{e}[0, \lambda']}[0, \lambda]$, so with $s = \bar{s}(\lambda) \in \mathcal{O}_e(\lambda)$, $\bar{s}[0, \lambda] \in \Sigma_{(s, e_\lambda)}$.

-

□

The following are some of the consequences of Thm 54.

Theorem 55. *If Z is an all-reet e -automata, $p, p_1, p_2 \in \text{Int}(Z)$, $e \in \mathcal{E}_F$, and $\lambda \in [0, \Lambda(\mathcal{O}_e)]$*

1) *If $(s_1, e_1), (s_2, e_2) \in \Theta_e(\lambda)$ then $(s_1, e_2) \in \Theta_e(\lambda)$*

2) *If $(s, e_1), (s, e_2) \in \Theta_e(\lambda)$ then $\Sigma_{(s, e_1)} = \Sigma_{(s, e_2)}$*

3) *If Z is boolean and all-reet and $(s, e_1), (s, e_2) \in \Theta_{[e]}(\lambda)$ then $\Sigma_{(s, e_1)} = \Sigma_{(s, e_2)}$*

Proof. 1 and 2 are immediate from Thm 54. 3 follows from Thm 54 and the definition of $[e]$. □

3. Ideal E-Automata

Definition 56. An e -automata is *ideal* if it is boolean and all-reet.

Ideal e -automata lack the ambiguities mentioned at the beginning of the section. The remainder of this section will seek to establish a central property ideal e -automata, to be given in Thm 59.

Theorem 57. *For ideal e -automata Z , for any $p_1, p_2 \in \text{Int}(Z)$*

1) *If $p_1 \cdot \mathcal{S} = p_2 \cdot \mathcal{S}$, and for some $e \in \mathcal{E}_F$, $\lambda \in [0, \Lambda(\mathcal{O}_e)]$, $p_1, p_2 \in \Theta_{[e]}(\lambda)$ then $\Sigma_{p_1} = \Sigma_{p_2}$, in all other cases $\Sigma_{p_1} \cap \Sigma_{p_2} = \emptyset$.*

2) *If $\Sigma_{p_1} = \Sigma_{p_2}$ then for all $e \in \mathcal{E}_F$, $\lambda \in [0, \Lambda(\mathcal{O}_e)]$, $p_1 \in \Theta_{[e]}(\lambda)$ iff $p_2 \in \Theta_{[e]}(\lambda)$*

Proof. 1) If the conditions hold then $\Sigma_{p_1} = \Sigma_{p_2}$ by Thm 55.3.

If $p_1 \cdot \mathcal{S} \neq p_2 \cdot \mathcal{S}$ then clearly $\Sigma_{p_1} \cap \Sigma_{p_2} = \emptyset$.

If there's doesn't exist a $e \in \mathcal{E}_F$ s.t. $p_1, p_2 \in \text{Ran}(\Theta_{[e]})$ then, since Z is boolean, $\Sigma_{p_1} \circ \Sigma_{p_1 \rightarrow} \cap \Sigma_{p_2} \circ \Sigma_{p_2 \rightarrow} = \emptyset$. By Thm 39, $\Sigma_{p_1} \cap \Sigma_{p_2} = \emptyset$

If there exists an $e \in \mathcal{E}_F$ s.t. $p_1, p_2 \in \text{Ran}(\Theta_{[e]})$, but no $\lambda \in [0, \Lambda(\mathcal{O}_e)]$ s.t. $p_1, p_2 \in \Theta_{[e]}(\lambda)$ then $\Sigma_{p_1} \cap \Sigma_{p_2} = \emptyset$ by Thm 52.

2) By Thm 39 $\Sigma_{p_1} \circ \Sigma_{p_1 \rightarrow} = \Sigma_{p_2} \circ \Sigma_{p_2 \rightarrow}$; since Z is boolean $p_1 \in \text{Ran}(\Theta_{[e]})$ iff $p_2 \in \text{Ran}(\Theta_{[e]})$. From Thm 52 it then follows that $p_1 \in \Theta_{[e]}(\lambda)$ iff $p_2 \in \Theta_{[e]}(\lambda)$. \square

For ideal e-automata, the sets $|\mathcal{O}_e[0, \lambda]|$ (see Section III B 3) are particularly useful; they tell you what has been measured as of λ .

Theorem 58. *If Z is an ideal e-automata then for every $e \in \mathcal{E}_F$, every $\lambda \in [0, \Lambda(\mathcal{O}_e)]$, every $e_\lambda \in (\Theta_{[e]} \cdot \mathcal{E})(\lambda)$, $|\mathcal{O}_e[0, \lambda]| = \bigcup_{s \in |\mathcal{O}_e[0, \lambda]|(\lambda)} \Sigma_{(s, e_\lambda)}$*

Proof. A: If Z is an ideal e-automata then for every $e \in \mathcal{E}_F$, every $\lambda \in [0, \Lambda(\mathcal{O}_e)]$, every $e_\lambda \in (\Theta_{[e]} \cdot \mathcal{E})(\lambda)$, $|\mathcal{O}_e[0, \lambda]| = \bigcup_{s \in \mathcal{O}_e(\lambda)} \Sigma_{(s, e_\lambda)}$

- Immediate from Thm 54 and Thm 55.3 -

It is sufficient to show that for every $n \in \mathbb{N}$, $|\mathcal{O}_e[0, \lambda]|_n = \bigcup_{s \in |\mathcal{O}_e[0, \lambda]|_n(\lambda)} \Sigma_{(s, e_\lambda)}$. By (A) this holds for $n = 0$. Assume it holds for $n = i$ and consider $n = i + 1$.

For any $\mathcal{O}_{e'}$ s.t. $|\mathcal{O}_{e'}[0, \lambda]| \cap |\mathcal{O}_e[0, \lambda]|_i \neq \emptyset$ for every $e_2 \in (\Theta_{[e']} \cdot \mathcal{E})(\lambda)$, $|\mathcal{O}_{e'}[0, \lambda]| = \bigcup_{s \in \mathcal{O}_{e'}(\lambda)} \Sigma_{(s, e_2)}$. By assumption $|\mathcal{O}_e[0, \lambda]|_i = \bigcup_{s \in |\mathcal{O}_e[0, \lambda]|_i(\lambda)} \Sigma_{(s, e_\lambda)}$, so by Thm 57.1 there's an $s \in |\mathcal{O}_e[0, \lambda]|_i(\lambda) \cap \mathcal{O}_{e'}(\lambda)$ s.t. $\Sigma_{(s, e_2)} = \Sigma_{(s, e_\lambda)}$; by Thm 57.2 $(s, e_\lambda) \in \Theta_{[e]}(\lambda)$, so by (A) $|\mathcal{O}_{e'}[0, \lambda]| = \bigcup_{s \in \mathcal{O}_{e'}(\lambda)} \Sigma_{(s, e_\lambda)}$. Since $|\mathcal{O}_e[0, \lambda]|_{i+1}$ is the union over all such $|\mathcal{O}_{e'}[0, \lambda]|$ it follows immediately that $|\mathcal{O}_e[0, \lambda]|_{i+1} = \bigcup_{s \in |\mathcal{O}_e[0, \lambda]|_{i+1}(\lambda)} \Sigma_{(s, e_\lambda)}$. \square

Theorem 59. *If Z is an ideal e-automata then for any $e \in \mathcal{E}_F$, $\lambda \in [0, \Lambda(\mathcal{O}_e)]$, $\mathcal{O}_e = |\mathcal{O}_e[0, \lambda]| \circ \mathcal{O}_e[\lambda, \Lambda(\mathcal{O}_e)]$*

Proof. With $e_\lambda \in (\Theta_e \cdot \mathcal{E})(\lambda)$:

$$\begin{aligned} \mathcal{O}_e &= \mathcal{O}_e[0, \lambda] \circ \mathcal{O}_e[\lambda, \Lambda(\mathcal{O}_e)] \text{ (Thm 53)} \\ &= (\bigcup_{s \in \mathcal{O}_e(\lambda)} \Sigma_{(s, e_\lambda)}) \circ \mathcal{O}_e[\lambda, \Lambda(\mathcal{O}_e)] \text{ (Thm 54)} \\ &= (\bigcup_{s \in |\mathcal{O}_e[0, \lambda]|(\lambda)} \Sigma_{(s, e_\lambda)}) \circ \mathcal{O}_e[\lambda, \Lambda(\mathcal{O}_e)] \\ &= |\mathcal{O}_e[0, \lambda]| \circ \mathcal{O}_e[\lambda, \Lambda(\mathcal{O}_e)] \text{ (Thm 58)} \end{aligned}$$

\square

D. Ideal Partitions

This section will be concerned with the necessary and sufficient conditions for a partition of a dynamic set to be the set of outcomes of an ideal e-automata. As a warm up, we'll start by picking up where we left off in Section III B 4 and consider the case where the system is a dynamic space; the more general case, where the system is a dynamic set, will be handled immediately thereafter. In order to attack these problems, several basic definitions first have to be given.

Definition 60. If S is a dynamic set, $\alpha \subset S$ is *bounded from above (by λ)* if $\alpha = \alpha[-\infty, \lambda] \circ S[\lambda, \infty]$, α is *bounded from below (by λ)* if $\alpha = S[-\infty, \lambda] \circ \alpha[\lambda, \infty]$, and α is *bounded* if it is both bounded from above and bounded from below.

If A is a set of subsets of S , A is *bounded from above/below (by λ)* every element of A is bounded from above/below by λ ; it's *bounded* if it's bounded from both above and below.

“Bounded from below by λ ” may be abbreviated “ $bb\lambda$ ” and “Bounded from above by λ ” may be abbreviated “ $ba\lambda$ ”.

Next to transfer the concept of $|\mathcal{O}_e[0, \lambda]|$ to partitions.

Definition 61. If K is a covering of dynamic set S , $\alpha \in \gamma$, $\lambda \in \Lambda_D$, and $x \leq \lambda$ is either $-\infty$ or an element of Λ_S

$$\begin{aligned} |\alpha[x, \lambda]|_0 &\equiv \alpha[x, \lambda] \\ |\alpha[x, \lambda]|_{n+1} &\equiv \{\bar{s}[x, \lambda] : \text{for some } \beta \in K \text{ s.t. } \beta[x, \lambda] \cap |\alpha[x, \lambda]|_n \neq \emptyset, \bar{s}[x, \lambda] \in \beta[x, \lambda]\} \\ |\alpha[x, \lambda]| &\equiv \bigcup_{n \in \mathbb{N}} |\alpha[x, \lambda]|_n \end{aligned}$$

By far the most important case is when $x = -\infty$; this is because of its use in defining ideal partitions. Ideal partitions will initially be defined just for dynamic spaces.

Definition 62. If γ a partition of dynamic space D , it is an *ideal partition (ip)* if it is bounded from below, all $\alpha \in \gamma$ are bounded from above, and for all $\alpha \in \gamma$, $\lambda \in \Lambda_S$, $\alpha = |\alpha[-\infty, \lambda]| \circ \alpha[\lambda, \infty]$.

The result to be attained in this section is that a covering is decidable by an ideal e-automata if and only if the covering is an ideal partition. The first step towards that result is the following theorem.

Theorem 63. *If a covering, K , of dynamic space D is decided by an ideal e -automata, $Z = (D_Z, I, F)$, then K is an ideal partition*

Proof. Since K is decided by Z , for some $\lambda_0 \in \Lambda_D$, $K \equiv \{\rightarrow \mathcal{O}_e^{\lambda_0} \rightarrow : e \in \mathcal{E}_F\}$. For each $\alpha = \rightarrow \mathcal{O}_e^{\lambda_0} \rightarrow \in K$ define $\lambda_\alpha \equiv \lambda_0 + \Lambda(\mathcal{O}_e)$.

A: K is $bb\lambda_0$:

- Immediate from the definition of $\rightarrow \mathcal{O}_e^{\lambda_0} \rightarrow$ -

B: Every $\alpha \in K$ is bounded from above

- Immediate from Thm 50 and $K = \{\rightarrow \mathcal{O}_e^{\lambda_0} \rightarrow : e \in \mathcal{E}_F\}$ -

C: K is a partition

- Immediate from Thm 44 -

D: For all $\lambda > \lambda_0$, $\alpha \in K$, $|\alpha[-\infty, \lambda]| = \rightarrow |\alpha[\lambda_0, \lambda]|$

- It is sufficient to show that for all $n \in \mathbb{N}$, $|\alpha[-\infty, \lambda]|_n = \rightarrow |\alpha[\lambda_0, \lambda]|_n$. By (A), this holds for $n = 0$. Assume it holds for $n = i$. By (A) and assumption on i , for all $\beta \in K$, $\beta[-\infty, \lambda] \cap |\alpha[-\infty, \lambda]|_i \neq \emptyset$ iff $\beta[\lambda_0, \lambda] \cap |\alpha[\lambda_0, \lambda]|_i \neq \emptyset$. Since $|\alpha[x, \lambda]|_{i+1}$ is equal to the union over the β 's that intersect $|\alpha[x, \lambda]|_i$, $|\alpha[-\infty, \lambda]|_i$ & $|\alpha[\lambda_0, \lambda]|_i$ are intersected by the same set of $\beta \in K$, and because all such β are $bb\lambda$, $|\alpha[-\infty, \lambda]|_{i+1} = \rightarrow |\alpha[\lambda_0, \lambda]|_{i+1}$ -

It remains to show that for all $\alpha \in K$, $\lambda \in \Lambda_D$, $\alpha = |\alpha[-\infty, \lambda]| \circ \alpha[\lambda, \infty]$. Clearly $\alpha \subset |\alpha[-\infty, \lambda]| \circ \alpha[\lambda, \infty]$, so it only needs to be shown that $|\alpha[-\infty, \lambda]| \circ \alpha[\lambda, \infty] \subset \alpha$.

For $\lambda \leq \lambda_0$: Immediate from (A).

For $\lambda \in (\lambda_0, \lambda_\alpha)$: By Thm 59, $\alpha[\lambda_0, \lambda_\alpha] = |\alpha[\lambda_0, \lambda]| \circ \alpha[\lambda, \lambda_\alpha]$. By (B) and (D) $|\alpha[-\infty, \lambda]| \circ \alpha[\lambda, \infty] = (\rightarrow |\alpha[\lambda_0, \lambda]|) \circ (\alpha[\lambda, \lambda_\alpha] \rightarrow) = \rightarrow (|\alpha[\lambda_0, \lambda]| \circ \alpha[\lambda, \lambda_\alpha]) \rightarrow = \rightarrow \alpha[\lambda_0, \lambda_\alpha] \rightarrow = \alpha$.

For $\lambda \geq \lambda_\alpha$: By (B) and (C), $|\alpha[-\infty, \lambda]| = \alpha[-\infty, \lambda]$, so $|\alpha[-\infty, \lambda]| \circ \alpha[\lambda, \infty] = \alpha[-\infty, \lambda] \circ \alpha[\lambda, \infty] \subset \alpha[-\infty, \lambda] \rightarrow = \alpha$. \square

Because ideal e -automata need only be weakly unbiased, it is insufficient to only consider measurements on dynamic spaces, so let's now to consider the more general case of a system that's a dynamic set.

The first thing to do is to extend the “outer \rightarrow ” notation to dynamic sets.

Definition 64. If S is a dynamic set and A is a set of partial paths then $\rightarrow A \rightarrow$, $\rightarrow A$, and $A \rightarrow$ are defined as before:

$\bar{p} \in \rightarrow A \rightarrow$ if there exists a $\bar{p}'[x_1, x_2] \in A$ s.t. $\bar{p} \in S[-\infty, x_1] \circ \bar{p}'[x_1, x_2] \circ S[x_2, \infty]$, etc.

$\rightarrow A \rightarrow$ is $\rightarrow A \rightarrow$ relativized to S : $\rightarrow A \rightarrow \equiv (\rightarrow A \rightarrow) \cap S$;

similarly for $\rightarrow A$ and $A \rightarrow$.

As a convenient shorthand, $+A \equiv \rightarrow A \rightarrow$.

If S is a dynamic space then $\rightarrow A = \rightarrow A$ and $A \rightarrow = A \rightarrow$.

It will also be useful to also extend the notion of boundedness to cover various situations that can arise with dynamic sets. (Note that the definition for bounded from above/below was given for dynamic sets, and so still holds.)

Definition 65. If S is a dynamic set and $\alpha \subset S$:

If $\alpha = \alpha[-\infty, \lambda] \rightarrow$ then α is *weakly bounded from above (by λ)*; if $\alpha = \rightarrow \alpha[\lambda, \infty]$ then α is *weakly bounded from below (by λ)*.

If for all $\lambda' \leq \lambda$, $\alpha = \rightarrow \alpha[-\infty, \lambda']$, α is *strongly bounded from below (by λ)*; similarly for *strongly bounded from above (by λ)*.

If A is a set of subsets of S , A is *strongly/weakly bounded from above/below (by λ)* every element of A is strongly/weakly bounded from above/below (by λ).

“Weakly bounded from above by λ ” may be abbreviated “*wba λ* ”, “strongly bounded from above by λ ” may be abbreviated “*sba λ* ”, etc.

If S is a dynamic space there is no difference between being bounded from above/below, weakly bounded from above/below, and strongly bounded from above/below.

Theorem 66. If α is *wba λ* and $\lambda' > \lambda$ then α is *wba λ'*

Proof. Assume $\bar{p}_1[-\infty, \lambda'] \in \alpha[-\infty, \lambda']$, $\bar{p}_2[\lambda', \infty] \in S[\lambda', \infty]$, and $\bar{p} = \bar{p}_1[-\infty, \lambda'] \circ \bar{p}_2[\lambda', \infty] \in S$. Since α is *wba λ* and $\bar{p}[-\infty, \lambda] = \bar{p}_1[-\infty, \lambda] \in \alpha[-\infty, \lambda]$, $\bar{p} \in \alpha$. \square

The most inclusive notion of decidability on a dynamic set would be: e-automata (D_Z, I, F) decides covering K on dynamic set S if for some $\lambda \in \Lambda_S$

- 1) For all $\bar{p}[\lambda, \lambda'] \in \mathcal{O}_F^\lambda$, $\bar{p}[\lambda, \lambda'] \in S[\lambda, \lambda']$
- 2) $K = \{+\mathcal{O}_e^\lambda : e \in \mathcal{E}_F\}$.

However, this would mean that the interactions between the system and the environment are only regulated while the experiment is taking place; before and after the experiment, any kind of interactions would be allowed, even those that would undermine the autonomy of the system. It is more sensible to assume that these interactions are always unbiased. If the system and environment don't interact outside of the experiment, which would be a natural

assumption, then it's certainly the case that the interactions are always unbiased. Because nothing is measured prior to λ , prior to λ there is no distinction between being weakly & strongly unbiased. From λ onward, Thm 37 ought to hold. This leads to the following with regard to the make up of a system:

Definition 67. If S is a dynamic set, K is a covering of S , and $\lambda \in \Lambda_S$, S is unbiased with respect to (λ, K) if

- 1) S is $sbb\lambda$
- 2) For all $a \in K$, all $\lambda' > \lambda$, all $\bar{p}_1, \bar{p}_2 \in +|a[-\infty, \lambda']|$, if $\bar{p}_1(\lambda') = \bar{p}_2(\lambda')$ then $\bar{p}_1[-\infty, \lambda'] \circ \bar{p}_2[\lambda', \infty] \in S$.

It may be assumed that K is $sbb\lambda$, but it is not demanded.

We're now in a position to define decidability in general.

Definition 68. A covering, K , of dynamic set S is *decided* by e-automata $Z = (D_Z, I, F)$ if $\Lambda_S = \Lambda_{D_Z}$ and for some $\lambda \in \Lambda_D$:

- 1) For all $\bar{p}[\lambda, \lambda'] \in \mathcal{O}_F^\lambda$, $\bar{p}[\lambda, \lambda'] \in S[\lambda, \lambda']$.
- 2) S is unbiased with respect to (λ, K) .
- 3) $K \equiv \{+\mathcal{O}_e^\lambda : e \in \mathcal{E}_F\}$.

Ideal partitions now also need to be generalized for dynamic sets.

Definition 69. If γ a partition of dynamic set S , it is an *ideal partition* (*ip*) if it is strongly bounded from below, all $\alpha \in \gamma$ are weakly bounded from above, and for all $\alpha \in \gamma$, $\lambda \in \Lambda_S$, $\alpha = |\alpha[-\infty, \lambda]| \circ \alpha[\lambda, \infty]$.

In the case where S is a dynamic set, this reduces to the prior definition. Now to generalize Thm 63.

Theorem 70. If a covering, K , of dynamic set S is decided by an ideal e-automata, $Z = (D_Z, I, F)$, then K is an ideal partition

Proof. Since K is decided by Z , for some $\lambda_0 \in \Lambda_S$, $K \equiv \{+\mathcal{O}_e^{\lambda_0} : e \in \mathcal{E}_F\}$. For each $\alpha = +\mathcal{O}_e^{\lambda_0} \in K$ define $\lambda_\alpha \equiv \lambda_0 + \Lambda(\mathcal{O}_e)$.

A: K is $sbb\lambda_0$

- Take any $\alpha \in K$, $\lambda \leq \lambda_0$; it is necessary to show that $\alpha = \rightarrow \alpha[\lambda, \infty]$. For any $\bar{p}_1[-\infty, \lambda] \in S[-\infty, \lambda]$, $\bar{p}_2[\lambda, \infty] \in \alpha[\lambda, \infty]$ s.t. $\bar{p}_1(\lambda) = \bar{p}_2(\lambda)$, $\bar{p} = \bar{p}_1[-\infty, \lambda] \circ \bar{p}_2[\lambda, \infty] \in S$

because S is unbiased with respect to (λ, K) . Since $\lambda \leq \lambda_0$, $\bar{p}_2[\lambda_0, \infty] \in \alpha[\lambda_0, \infty]$, so since $\alpha = +\alpha[\lambda_0, \lambda_\alpha]$, $\bar{p} \in \alpha$.

B: Every $\alpha \in K$ is weakly bounded from above

-Immediate from Thm 50 and $K = \{\neg \mathcal{O}_e^{\lambda_0} \neg : e \in \mathcal{E}_F\}$ -

C: K is a partition

- Follows from the boolean & automata conditions -

D: For all $\lambda \in \Lambda_S$, $\alpha \in K$, $\alpha[-\infty, \lambda] \circ \alpha[\lambda, \infty] \subset S$

- Follows from S being unbiased with respect to (λ, K) . -

E: For all $\lambda > \lambda_0$, $\alpha \in K$, $|\alpha[-\infty, \lambda]| \Rightarrow |\alpha[\lambda_0, \lambda]|$

- Identical to (D) in Thm 63 -

It remains to show that for all $\alpha \in K$, $\lambda \in \Lambda_S$, $\alpha = |\alpha[-\infty, \lambda]| \circ \alpha[\lambda, \infty]$. As before, $\alpha \subset |\alpha[-\infty, \lambda]| \circ \alpha[\lambda, \infty]$, so it only needs to be shown that $|\alpha[-\infty, \lambda]| \circ \alpha[\lambda, \infty] \subset \alpha$.

For $\lambda \leq \lambda_0$: Immediate from (A).

For $\lambda \in (\lambda_0, \lambda_\alpha)$: Assume $\bar{p}_1[-\infty, \lambda] \in |\alpha[-\infty, \lambda]|$ and $\bar{p}_2[\lambda, \infty] \in \alpha[\lambda, \infty]$. It follows from (E) that $\bar{p}_1[\lambda_0, \lambda] \in |\alpha[\lambda_0, \lambda]|$, and so from Thm 59 that $\bar{p}_1[\lambda_0, \lambda] \circ \bar{p}_2[\lambda, \lambda_\alpha] \in \alpha[\lambda_0, \lambda_\alpha]$. It then follows from (A) that $\bar{p}_1[-\infty, \lambda] \circ \bar{p}_2[\lambda, \lambda_\alpha] \in \alpha[-\infty, \lambda_\alpha]$. Since S is unbiased with respect to (λ, K) , $\bar{p}_1[-\infty, \lambda] \circ \bar{p}_2[\lambda, \infty] \in S$. Therefore, since $\alpha = \alpha[-\infty, \lambda_\alpha] \neg$, $\bar{p}_1[-\infty, \lambda] \circ \bar{p}_2[\lambda, \lambda_\alpha] \in \alpha$.

For $\lambda \geq \lambda_\alpha$: By (B) and (C), $|\alpha[-\infty, \lambda]| = \alpha[-\infty, \lambda]$, and by (D) $\alpha[-\infty, \lambda] \circ \alpha[\lambda, \infty] \subset \alpha[-\infty, \lambda] \neg$. By (B) $\alpha = \alpha[-\infty, \lambda] \neg$. \square

And now establish the inverse, that all ip's are ideally decidable.

Theorem 71. *All ideal partitions are ideally decidable*

Proof. Assume γ is an ideal partition of dynamic set S . An ideal e-automata, $Z = (D, I, F)$, that decides γ will be constructed.

1) Start by constructing a new dynamic space $D_0 = S \otimes E_0$ as follows: For every $\lambda \in \Lambda_D$ create a set $E(\lambda)$, and a bijection $b_\lambda : |\gamma[-\infty, \lambda]| \rightarrow E(\lambda)$ s.t. for all $\lambda_1 \neq \lambda_2$, $E(\lambda_1) \cap E(\lambda_2) = \emptyset$; $(\bar{s}, \bar{e}) \in D_0$ iff $\bar{s} \in S$ and for all $\lambda \in \Lambda$, if $\bar{s} \in \alpha \in \gamma$ then $\bar{e}(\lambda) = b_\lambda(|\alpha[-\infty, \lambda]|)$

2) From D_0 construct Z 's dynamic space, D :

If Λ_S is unbounded from below, for every $\lambda \in \Lambda_D$ define $D_\lambda \equiv \{\bar{p} : \text{for some } \bar{p}' \in D_0, \text{ for all } \lambda' \in \Lambda_S, \bar{p}'(\lambda' + \lambda) = \bar{p}(\lambda')\}$. $D_Z \equiv \bigcup_{\lambda \in \Lambda} D_\lambda$.

If Λ_D is bounded from below, take D_0^* to be any dynamic space s.t. $\Lambda_{D_0^*}$, is unbounded from below, $D_0^*[0, \infty] = D_0$, and for all $\lambda < \lambda' < 0$, $D_0^*(\lambda) \cap D_0^*(\lambda') = \emptyset$ (this already holds for all $\lambda > \lambda' \geq 0$). Construct D^* as above and take $D = D^*[0, \infty]$.

3) Construct I : Select any $\lambda_0 \in \Lambda_D$ s.t. γ is $bb\lambda_0$, $I \equiv D_0(\lambda_0)$

4) Construct F : For every $\alpha \in \gamma$ select a $\lambda_\alpha \in \Lambda_D$ s.t. $\alpha = +\alpha[-\infty, \lambda_\alpha]$; $F \equiv \bigcup_{\alpha \in \gamma} \alpha(\lambda_\alpha) \otimes \{b_{\lambda_\alpha}(|\alpha[-\infty, \lambda_\alpha]|)\}$.

Now to show that (D, I, Z) is an ideal e-automata that decides γ .

A: D_0 is a dynamic space

- Take any $(\bar{s}_1, \bar{e}_1), (\bar{s}_2, \bar{e}_2) \in D_0$ s.t. for some $\lambda \in \Lambda_S$, $(\bar{s}_1(\lambda), \bar{e}_1(\lambda)) = (\bar{s}_2(\lambda), \bar{e}_2(\lambda))$.

Take $\bar{s}_1 \in \alpha_1 \in \gamma$ and $\bar{s}_2 \in \alpha_2 \in \gamma$; note that since $\bar{e}_1(\lambda) = \bar{e}_2(\lambda)$, $|\alpha_1[-\infty, \lambda]| = |\alpha_2[-\infty, \lambda]|$.

Take $(\bar{s}, \bar{e}) \equiv (\bar{s}_1[-\infty, \lambda] \circ \bar{s}_2[\lambda, \infty], \bar{e}_1[-\infty, \lambda] \circ \bar{e}_2[\lambda, \infty])$. $\bar{s} \in |\alpha_1[-\infty, \lambda]| \circ \alpha_2[\lambda, \infty] = |\alpha_2[-\infty, \lambda]| \circ \alpha_2[\lambda, \infty] = \alpha_2$, so $(\bar{s}, \bar{e}) \in D_0$ iff for all $\lambda' \in \Lambda_D$, $\bar{e}(\lambda') = b_{\lambda'}(|\alpha_2[-\infty, \lambda']|)$, which follows from the fact that $\bar{s} \in \alpha_2$. -

B: D is a dynamic space

- For any $p \in \mathcal{P}_{D_0} = \mathcal{P}_D$, p is only realized at a single $\lambda \in \Lambda_S$ in D_0 . Because D is a union over “shifted” copies of D_0 , paths in D can only intersect if they belong to the same copy; it follows that D is a dynamic space if D_0 is. -

C: D is homogeneous; I is homogeneously realized

- Because for any $p \in \mathcal{P}_{D_0}$, p is only realized at a single $\lambda \in \Lambda_S$ in D_0 , D_0 must be homogeneous. From the nature of the construction of D from D_0 it's clear that D is also homogeneous, and that all of \mathcal{P}_{D_0} is homogeneously realized in D . -

D: $I \rightarrow = I \upharpoonright (I \rightarrow F) \rightarrow$

$\rightarrow F = \rightarrow I \upharpoonright (I \rightarrow F)$

- $D_0 = \rightarrow I \upharpoonright I \rightarrow F \rightarrow$ and so for D_0 , $I \rightarrow = I \upharpoonright (I \rightarrow F) \rightarrow$ and $\rightarrow F = \rightarrow I \upharpoonright (I \rightarrow F)$.

If this holds for D_0 , it must also hold for D -

E: Z is an environmental shell

- All elements of $Dom(Z)$ can be decomposed into system & environmental states, and F has its own set of environmental states -

F: Z is weakly unbiased

- Recall that $I = D_0(\lambda_0)$. Choose any $e_1 \in \mathcal{E}_F$, and take $\alpha \in \gamma$, $\lambda_1 + \lambda_0 \in \Lambda_S$ s.t. $b_{\lambda_0 + \lambda_1}(|\alpha[-\infty, \lambda_0 + \lambda_1]|) = e_1$ (since $e_1 \in \mathcal{E}_F$ there can only be one such α). Take any $e \in \mathcal{E}_{Int(Z)}$, $\lambda \in \Lambda_D$ s.t. $\Sigma_e \cap |\mathcal{O}_{e_1}[0, \lambda]| \neq \emptyset$. It follows that $e = b_{\lambda_0 + \lambda}(|\alpha[-\infty, \lambda_0 + \lambda]|)$.

Therefore, if $e, e' \in \mathcal{E}_{Int(Z)}$ and for some $\lambda \in \Lambda$, $\Sigma_e \cap |\mathcal{O}_{e_1}[0, \lambda]| \neq \emptyset$ and $\Sigma_{e'} \cap |\mathcal{O}_{e_1}[0, \lambda]| \neq \emptyset$ then $e = e'$. Z must then be weakly unbiased -

G: Z is an e-automata

- Take any $\bar{s}_1[0, \lambda_1], \bar{s}_2[0, \lambda_2] \in \mathcal{O}_F$ s.t. $\lambda_2 \geq \lambda_1$ and $\bar{s}_1[0, \lambda_1] = \bar{s}_2[0, \lambda_1]$. For $\bar{p}_1[0, \lambda_1], \bar{p}_2[0, \lambda_2] \in \Theta_F$ s.t. $\bar{p}_1[0, \lambda_1] \cdot \mathcal{S} = \bar{s}_1[0, \lambda_1]$ and $\bar{p}_2[0, \lambda_2] \cdot \mathcal{S} = \bar{s}_2[0, \lambda_2]$, $\bar{p}_1(\lambda_1) = \bar{p}_2(\lambda_1) \in F$, so $\lambda_2 = \lambda_1$. -

H: Z is all-reet

- For any $e \in \mathcal{E}_F$ there's only a single $\bar{e}[0, \lambda] \in \omega_F \cdot \mathcal{E}$, so naturally $\mathcal{O}_e = \mathcal{O}_{\bar{e}[0, \lambda]}$ -

I: Z is boolean

- For any $\bar{s}[0, \lambda] \in \mathcal{O}_F$ there's only one $(\bar{s}[0, \lambda], \bar{e}[0, \lambda]) \in \Theta_F$. Since $\bar{e}(\lambda) \in \mathcal{E}_F$ there's only one $e \in \mathcal{E}_F$ s.t. $\bar{s}[0, \lambda] \in \mathcal{O}_e$ -

J: Z decides γ

- Given any $\alpha \in \gamma$, take $e = b_{\lambda_\alpha}(|\alpha[-\infty, \lambda_\alpha]|)$; $e \in \mathcal{E}_F$ and $\alpha = +\mathcal{O}_e^{\lambda_0}$ - \square

Note that S need not be homogeneous, nor does any part of \mathcal{P}_S need to be homogeneously realized.[3]

Theorem 72. *If γ is an ip and $\bigcup \gamma$ is a dynamic space, then γ is decided by a strongly unbiased ideal e-automate.*

Proof. Use the same construction as the prior theorem. For every $(s, e) \in D_0$, $e = b_\lambda(|\alpha[-\infty, \lambda]|)$, $(D_0)_{(s, e) \rightarrow} \cdot \mathcal{S} = (|\alpha[-\infty, \lambda]|)_{(\lambda, s) \rightarrow}$. If $\bigcup \gamma$ is a dynamic space, this means that $(D_0)_{(s, e) \rightarrow} \cdot \mathcal{S} = (\bigcup \gamma)_{(\lambda, s) \rightarrow}$. Since $\Sigma_{(s, e) \rightarrow}$ is simply $(D_0)_{(s, e) \rightarrow} \cdot \mathcal{S}$ shifted by λ_0 and truncated at the λ_α 's (for definitions of λ_0 and λ_α , see (3) and (4) in the proof of the prior theorem), it follows immediately that the e-automata is strongly unbiased. \square

E. Companionable Sets & Compatible Sets

In the final section of this part, the make-up of ip's will be investigated.

Definition 73. If S is a dynamic set:

$A \subset S$ is a *subspace* (of S) if it is non-empty and for every $\lambda \in \Lambda_S$, $A = A[-\infty, \lambda] \circ A[\lambda, \infty]$.

$A \subset S$ is *companionable* if it is a subspace that's weakly bounded from above and strongly bounded from below.

Theorem 74. *If γ is an ip and $\alpha \in \gamma$ then α is companionable*

Proof. For any $\lambda \in \Lambda$, $\alpha \subset \alpha[-\infty, \lambda] \circ \alpha[\lambda, \infty] \subset |\alpha[-\infty, \lambda]| \circ \alpha[\lambda, \infty]$. Since $\alpha = |\alpha[-\infty, \lambda]| \circ \alpha[\lambda, \infty]$, $\alpha = \alpha[-\infty, \lambda] \circ \alpha[\lambda, \infty]$.

By the definition of an ip, α must be strongly bounded from below and weakly bounded from above. \square

Definition 75. If S is a dynamic set, Γ is a non-empty set of subsets of S , and $\alpha \in \Gamma$:

$|\alpha[-\infty, \lambda]|_\Gamma$ is defined identically to how it's defined for covering in definition 61

$(\alpha)_\lambda^\Gamma \equiv \{\beta \in \Gamma : \beta[-\infty, \lambda] \subset |\alpha[-\infty, \lambda]|_\Gamma\}$.

Γ is *compatible* if it is a pairwise disjoint set of companionable sets, strongly bounded from below, and for all $\alpha, \beta \in \Gamma$, all $\lambda \in \Lambda_S$, if $\beta \in (\alpha)_\lambda^\Gamma$ and $p \in \alpha(\lambda) \cap \beta(\lambda)$ then $\alpha_{\rightarrow(\lambda, p)} = \beta_{\rightarrow(\lambda, p)}$.

If the set Γ is understood, $|\alpha[-\infty, \lambda]|_\Gamma$ may be written $|\alpha[-\infty, \lambda]|$ and $(\alpha)_\lambda^\Gamma$ may be written $(\alpha)_\lambda$.

Theorem 76. *If S is a dynamic set, Γ is a non-empty set of subsets of S , and $\alpha, \beta \in \Gamma$, then*

1) $\beta \in (\alpha)_\lambda^\Gamma$ iff there's a finite sequence of elements of Γ , $(a_i)_{i \leq n}$, s.t. $a_1 = \alpha$, $a_n = \beta$, and for all $1 \leq i < n$, $a_i[-\infty, \lambda] \cap a_{i+1}[-\infty, \lambda] \neq \emptyset$.

2) $\beta \in (\alpha)_\lambda^\Gamma$ iff $|\beta[-\infty, \lambda]|_\Gamma \cap |\alpha[-\infty, \lambda]|_\Gamma \neq \emptyset$ iff $|\beta[-\infty, \lambda]|_\Gamma = |\alpha[-\infty, \lambda]|_\Gamma$.

Proof. Immediate from the definition of $|\alpha[-\infty, \lambda]|_\Gamma$. \square

Theorem 77. Γ is compatible iff it's pairwise disjoint, strongly bounded from below, all $\alpha \in \Gamma$ are weakly bounded from above, and for all $\lambda \in \Lambda$, $\alpha = |\alpha[-\infty, \lambda]|_\Gamma \circ \alpha[\lambda, \infty]$.

Proof. \Rightarrow $|\alpha[-\infty, \lambda]|_\Gamma = \bigcup_{\beta \in (\alpha)_\lambda^\Gamma} \beta[-\infty, \lambda]$. Since Γ is compatible, for all $\beta \in (\alpha)_\lambda^\Gamma$, $p \in \alpha(\lambda) \cap \beta(\lambda)$, $\beta_{\rightarrow(\lambda, p)} = \alpha_{\rightarrow(\lambda, p)}$. Therefore $\alpha[-\infty, \lambda] \circ \alpha[\lambda, \infty] = |\alpha[-\infty, \lambda]|_\Gamma \circ \alpha[\lambda, \infty]$. Since α is companionable, $\alpha = \alpha[-\infty, \lambda] \circ \alpha[\lambda, \infty]$.

\Leftarrow For $\alpha, \beta \in \Gamma$, if $\beta \in (\alpha)_\lambda^\Gamma$ then $|\alpha[0, \lambda]| = |\beta[0, \lambda]|$, so $\beta = |\alpha[-\infty, \lambda]| \circ \beta[\lambda, \infty]$, and so for any $p \in \alpha(\lambda) \cap \beta(\lambda)$, $\alpha_{\rightarrow(\lambda, p)} = \beta_{\rightarrow(\lambda, p)}$.

$\alpha \subset \alpha[-\infty, \lambda] \circ \alpha[\lambda, \infty] \subset |\alpha[-\infty, \lambda]| \circ \alpha[\lambda, \infty] = \alpha$, so α is companionable. \square

Theorem 78. *A covering of a dynamic set is an ip iff it's compatible*

Proof. Immediate from Thm 77. □

This establishes that if α is the element of some ip then it's companionable, and if t is the subset of some ip then it's compatible. For dynamic spaces, the converse also holds. For any compatible set, t , an “all-reet not t ” can be constructed by taking the paths not in t and grouping them by when they broke off from t , and which $|\alpha[-\infty, \lambda]|$ they broke off from. t together with the “all-reet not t ” form an ip. When the parameter is not discrete one mild complication is that, in dealing with paths that broke off from t at λ , we'll need to distinguish between paths that were in t at λ , but aren't in t at any $\lambda' > \lambda$ from those that are not in t at λ , but were in t at all $\lambda' < \lambda$.

Definition 79. If t is compatible, S is a dynamic set s.t. $\bigcup t \subset S$, and $\alpha \in t$ then

$$\begin{aligned} \sim t_\lambda &\equiv \{\bar{p} \in S : \bar{p}[-\infty, \lambda] \notin \bigcup t[-\infty, \lambda]\} \\ \sim |\alpha|_\lambda^+ &\equiv \{\bar{p} \in S : \bar{p}[-\infty, \lambda] \in |\alpha[-\infty, \lambda]| \text{ and for all } \lambda' > \lambda, \bar{p}[-\infty, \lambda'] \notin \bigcup t[-\infty, \lambda']\} \\ \sim |\alpha|_\lambda^- &\equiv \{\bar{p} \in S : \bar{p}[-\infty, \lambda] \notin \bigcup t[-\infty, \lambda] \text{ and for all } \lambda' < \lambda, \bar{p}[-\infty, \lambda'] \in |\alpha[-\infty, \lambda']|\} \end{aligned}$$

These sets can be used to construct an ip containing t .

Theorem 80. If D is a dynamic space, α is a subset of D , and t is a set of subsets of D then

- 1) If α is companionable then there exists an ip, γ , s.t. $\alpha \in \gamma$
- 2) If t is compatible then there exists an ip, γ , s.t. $t \subset \gamma$

Proof. (1) Follows immediately from (2). For (2), assume t is $bb\lambda$ and take γ to be the set s.t. $t \subset \gamma$, if $\sim t_\lambda \neq \emptyset$ then $\sim t_\lambda \in \gamma$, and for all $\alpha \in t$, $\lambda' > \lambda$, if $\sim |\alpha|_{\lambda'}^{+/-} \neq \emptyset$ then $\sim |\alpha|_{\lambda'}^{+/-} \in \gamma$. γ is then partition of D . (From here on, the qualifiers “if $\sim t_\lambda \neq \emptyset$ ” and “if $\sim |\alpha|_{\lambda'}^{+/-} \neq \emptyset$ ” will be understood.)

The following is key to establishing that γ is an ip:

A: For all $\lambda_1 < \lambda_2$, $p \in \sim |\alpha|_{\lambda_2}^-$, $(\sim |\alpha|_{\lambda_2}^-)_{\rightarrow(\lambda_1, p)} = |\alpha[-\infty, \lambda_2]|_{\rightarrow(\lambda_1, p)}$

- By the definition of $\sim |\alpha|_\lambda^-$ it's clearly the case that $(\sim |\alpha|_{\lambda_2}^-)_{\rightarrow(\lambda_1, p)} \subset |\alpha[-\infty, \lambda_2]|_{\rightarrow(\lambda_1, p)}$. Take any $\bar{p}_1 \in +|\alpha[-\infty, \lambda_2]|$ and any $\bar{p}_2 \in \sim |\alpha|_{\lambda_2}^-$ s.t. $\bar{p}_1(\lambda_1) = \bar{p}_2(\lambda_1) = p$. Take $\bar{p} \equiv \bar{p}_1[-\infty, \lambda_1] \circ \bar{p}_2[\lambda_1, \infty] \in D$ and note that for all $\lambda' < \lambda_2$ $\bar{p}[-\infty, \lambda'] \in |\alpha[-\infty, \lambda']|$. Assume that for some $\beta \in t$, $\bar{p}[-\infty, \lambda_2] \in \beta[-\infty, \lambda_2]$. That would mean $|\beta[-\infty, \lambda_1]| = |\alpha[-\infty, \lambda_1]|$ and so $\bar{p}_2[-\infty, \lambda_1] \circ \bar{p}[\lambda_1, \lambda_2] = \bar{p}_2[-\infty, \lambda_2] \in \beta[-\infty, \lambda_2]$. This contradicts $\bar{p}_2 \in \sim |\alpha|_{\lambda_2}^-$, so

there can be no such $\beta \in t$. Therefore $\bar{p} \in \sim |\alpha|_{\lambda_2}^-$, and so for all $\lambda_1 < \lambda_2$, $(\sim |\alpha|_{\lambda_2}^-)_{\rightarrow(\lambda_1, p)} = |\alpha|_{[-\infty, \lambda_2]}|_{\rightarrow(\lambda_1, p)}$.

Similarly, for all $\lambda_1 \leq \lambda_2$, $p \in \sim |\alpha|_{\lambda_2}^+$, $(\sim |\alpha|_{\lambda_2}^+)_{\rightarrow(\lambda_1, p)} = |\alpha|_{[-\infty, \lambda_2]}|_{\rightarrow(\lambda_1, p)}$

For any $\lambda' > \lambda$ the following properties therefore hold for $\sim |\alpha|_{\lambda'}^-$ and $\sim |\alpha|_{\lambda'}^+$:

$\sim |\alpha|_{\lambda'}^-$ is $ba\lambda'$ and for all $\lambda_1 < \lambda'$, $p \in \sim |\alpha|_{\lambda'}^-(\lambda_1)$, $(\sim |\alpha|_{\lambda'}^-)_{\rightarrow(\lambda_1, p)} = |\alpha|_{[-\infty, \lambda']}|_{\rightarrow(\lambda_1, p)}$.

Similarly, for all $\lambda'' > \lambda'$, $\sim |\alpha|_{\lambda'}^+$ is $ba\lambda''$ and for all $\lambda_1 \leq \lambda'$, $p \in \sim |\alpha|_{\lambda'}^+(\lambda_1)$, $(\sim |\alpha|_{\lambda'}^+)_{\rightarrow(\lambda_1, p)} = |\alpha|_{[-\infty, \lambda']}|_{\rightarrow(\lambda_1, p)}$.

As an aside, it follows that the $\sim |\alpha|_{\lambda'}^{+/-}$ are $bb\lambda$.

Finally, since t is $bb\lambda$, $\sim t_\lambda = \rightarrow(\lambda, D(\lambda) - \bigcup t(\lambda)) \rightarrow$.

These properties are sufficient to establish that γ is an ip. □

This result may be generalized for systems that are not dynamic spaces.

Theorem 81. *If S is a dynamic set and γ is an ip of S then*

- 1) *If $\alpha \subset \beta \in \gamma$ and α is companionable then there exists an ip of S , γ' , s.t. $\alpha \in \gamma'$*
- 2) *If t is a compatible set of subsets of S s.t. for each $\alpha \in t$ there's a $\alpha \subset \beta \in \gamma$ then there exists an ip of S , γ' , s.t. $t \subset \gamma'$*

Proof. Once again, (1) follows from (2).

Assume t is $bb\lambda$ and this time take γ' to be the set s.t. $t \subset \gamma'$ and for all $\beta \in \gamma$ if $\beta \cap (\sim t_\lambda) \neq \emptyset$ then $\beta \cap (\sim t_\lambda) \in \gamma'$, and for all $\alpha \in t$, $\lambda' > \lambda$, if $\beta \cap (\sim |\alpha|_{\lambda'}^{+/-}) \neq \emptyset$ then $\beta \cap (\sim |\alpha|_{\lambda'}^{+/-}) \in \gamma'$.

The nature of these sets are similar to those of the prior proof, except that they're relativized to each $\beta \in \gamma$. The following properties are sufficient to establish that γ' is an ip:

For all $\lambda'' < \lambda'$, $p \in (\beta \cap (\sim |\alpha|_{\lambda'}^-))(\lambda'')$, $(\beta \cap (\sim |\alpha|_{\lambda'}^-))_{\rightarrow(\lambda'', p)} = |\alpha|_{[-\infty, \lambda']}|_{\rightarrow(\lambda'', p)}$ (Assume $\alpha \subset \eta \in \gamma$; since $\beta \cap (\sim |\alpha|_{\lambda'}^-) \neq \emptyset$, and $\lambda'' < \lambda'$, $\beta[-\infty, \lambda''] \subset |\eta|_{[-\infty, \lambda'']}$, so $\beta_{\rightarrow(\lambda'', p)} = |\eta|_{[-\infty, \lambda']}|_{\rightarrow(\lambda'', p)}$, and so $|\alpha|_{[-\infty, \lambda']}|_{\rightarrow(\lambda'', p)} \subset \beta_{\rightarrow(\lambda'', p)}$.)

$\beta \cap (\sim |\alpha|_{\lambda'}^-) = (\beta \cap (\sim |\alpha|_{\lambda'}^-))[-\infty, \lambda'] \circ \beta[\lambda', \infty]$.

For all $\lambda'' \leq \lambda'$, $p \in (\beta \cap (\sim |\alpha|_{\lambda'}^+))(\lambda'')$, $(\beta \cap (\sim |\alpha|_{\lambda'}^+))_{\rightarrow(\lambda'', p)} = |\alpha|_{[-\infty, \lambda']}|_{\rightarrow(\lambda'', p)}$

For all $\lambda'' > \lambda'$, $\beta \cap (\sim |\alpha|_{\lambda'}^+) = (\beta \cap (\sim |\alpha|_{\lambda'}^+))[-\infty, \lambda''] \circ \beta[\lambda'', \infty]$.

With $X \equiv [\beta \cap (\sim t_\lambda)](\lambda)$, $\beta \cap (\sim t_\lambda) = \beta_{\rightarrow(\lambda, X)} \circ \beta_{(\lambda, X) \rightarrow}$. □

The first theorem follows from the second, since if D is a dynamic space then $\{D\}$ is an ip.

IV. PROBABILITIES

A. Dynamic Probability Spaces

For a single ip, probabilities are no different than in classic probability and statistics[4]:

Definition 82. An *ip probability space* is a triple, (γ, Σ, P) , where γ is an ip, Σ is a set of subsets of γ , and $P : \Sigma \rightarrow [0, 1]$ s.t.:

- 1) $\gamma \in \Sigma$
- 2) If $\sigma \in \Sigma$ then $\gamma - \sigma \in \Sigma$
- 3) If $\psi \subset \Sigma$ is finite then $\bigcup \psi \in \Sigma$
- 4) $P(\gamma) = 1$
- 5) If $\psi \subset \Sigma$ is finite and pairwise disjoint then $P(\bigcup \psi) = \sum_{\sigma \in \psi} P(\sigma)$

Very commonly, “finite” in (3) and (5) is replaced with “countable”. Countable additivity is invaluable when dealing with questions of convergence, however convergence will become a more multifaceted issue in the structures to be introduced, so the countable condition has been relaxed & questions of convergence delayed.

Theorem 83. *In the definition of an ip probability space, (2) and (3) can be replaced with:*
If $\sigma_1, \sigma_2 \in \Sigma$ then $\sigma_1 - \sigma_2 \in \Sigma$

Proof. If (2) holds then (3) is equivalent to “If $\sigma_1, \sigma_2 \in \Sigma$ then $\sigma_1 \cap \sigma_2 \in \Sigma$ ”. The equivalence now follows from $\sigma_1 - (\sigma_1 - \sigma_2) = \sigma_1 \cap \sigma_2$ and $\sigma_1 - \sigma_2 = \sigma_1 \cap (\gamma - \sigma_2)$. \square

Definition 84. If S_1 and S_2 are dynamic spaces and γ_1 and γ_2 are ip’s of S_1 and S_2 respectively, then ip probability spaces $(\gamma_1, \Sigma_1, P_1)$ and $(\gamma_2, \Sigma_2, P_2)$ are *consistent* if

- 1) $\gamma_1 \cap \gamma_2 \in \Sigma_1$ and $\gamma_1 \cap \gamma_2 \in \Sigma_2$
- 2) If $\sigma \subset \gamma_1 \cap \gamma_2$ then $\sigma \in \Sigma_1$ iff $\sigma \in \Sigma_2$
- 3) For any $t \in \Sigma_1 \cap \Sigma_2$, $P_1(t) = P_2(t)$

If Y is a set of ip probability spaces, Y is *consistent* if, for any $x, y \in Y$, x and y are consistent.

A dynamic probability space is, essentially, a consistent collection of ip probability spaces.

Definition 85. A *dynamic probability space* (dps) is a triple (X, T, P) where X is a set of dynamic sets, T is a set of compatible sets, and $P : T \rightarrow [0, 1]$ s.t.:

- 1) For every $S \in X$ there's a $\gamma \in T$ s.t. γ is an ip of S
- 2) For every $t \in T$ there's a $\gamma \in T$ s.t. γ is an ip of some $S \in X$ and $t \subset \gamma$
- 3) If $t_1, t_2 \in T$ then $t_1 - t_2 \in T$
- 4) If $\gamma \in T$ is an ip of some $S \in X$ then $P(\gamma) = 1$
- 5) If $t_1, t_2 \in T$ are disjoint, and $t_1 \cup t_2 \in T$ then $P(t_1 \cup t_2) = P(t_1) + P(t_2)$

For dynamic probability space (X, T, P)

$$G_T \equiv \{\gamma \in T : \gamma \text{ is a partition of some } S \in X\}$$

If S is a dynamic space and $\alpha, \beta \subset S$, it's important to stress that axiom 5 means that $P(\{\alpha, \beta\}) = P(\{\alpha\}) + P(\{\beta\})$ (assuming $\{\alpha, \beta\}, \{\alpha\}, \{\beta\} \in T$); it does not mean that $P(\{\alpha, \beta\}) = P(\{\alpha \cup \beta\})$ (even if $\{\alpha, \beta\}, \{\alpha \cup \beta\} \in T$), and so in general $P(\{\alpha \cup \beta\}) \neq P(\{\alpha\}) + P(\{\beta\})$.

Note that Thm 83 does not hold for dps's. This is essentially because there is no $Z \in T$ s.t. for all $t \in T$, $t \subset Z$; as a result, "not t " is not uniquely defined, and arbitrary finite unions can not be expected to be elements of T (though arbitrary finite intersections are elements of T).

To formally describe the connection between dynamic probability spaces and classic probability theory, the following definitions will be useful.

Definition 86. If (X, T, P) is a dps and $\gamma \in G_T$ then $T_\gamma \equiv \{t \in T : t \subset \gamma\}$ and $P_\gamma \equiv P|_{T_\gamma}$ (that is, $\text{Dom}(P_\gamma) = T_\gamma$ and for all $t \in T_\gamma$, $P_\gamma(t) = P(t)$)

If $\pi = (X, T, P)$ is a dps and $A \subset G_T$, $A_\pi \equiv \{(\gamma, T_\gamma, P_\gamma) : \gamma \in A\}$

If Y is a consistent set of ip probability spaces, $X_Y \equiv \{\bigcup \gamma : (\gamma, \Sigma, P) \in Y\}$, $T_Y \equiv \bigcup_{(\gamma, \Sigma, P) \in Y} \Sigma$ and $P_Y : T_Y \rightarrow [0, 1]$ s.t. if $(\gamma, \Sigma, P) \in Y$ and $t \in \Sigma$ then $P_Y(t) = P(t)$.

The following theorem states that a dynamic probability space is simply a consistent set of ip probability spaces.

Theorem 87. 1) If $\pi = (X, T, P)$ is a dps and $A \subset G_T$ then A_π is a consistent set of ip probability spaces

2) If Y is a consistent set of ip probability spaces then (X_Y, T_Y, P_Y) is dps

3) If (X, T, P) is a dps and $Y \equiv \{(\gamma, T_\gamma, P_\gamma) : \gamma \in G_T\}$ then $X = X_Y$, $T = T_Y$ and $P = P_Y$

4) If Y is a consistent set of ip probability spaces then $Y = \{(\gamma, (T_Y)_\gamma, (P_Y)_\gamma) : \gamma \in G_{(T_Y)}\}$

Proof. (1) says that a dps can be rewritten as a consistent set of ip probability spaces, (2) says that a consistent set of ip probability spaces can be rewritten as a dps, and (3) & (4) say that in moving between dps's & consistent sets of ip probability spaces, no information is lost.

A: For dps (X, T, P) , if $t_1, t_2 \in T$ and γ is any element of G_T s.t. $t_1 \subset \gamma$ then $t_1 - t_2 \in T_\gamma$ - Follows from $t_1 - t_2 = t_1 - t_1 \cap t_2 = t_1 - \gamma \cap t_2$ (note that $\gamma \cap t_2 \in T_\gamma$) -

(1) That the elements of A_π are ip probability spaces follows from Thm 83. That they're consistent follows from $\gamma_1 \cap \gamma_2 = \gamma_1 - (\gamma_1 - \gamma_2)$

(2) Axioms 1, 2, and 4 clearly hold. Axiom 3 follows from Thm 83 and (A). Axiom 5 follows from axiom 2.

(3) & (4) That no information is lost in going from a dps to a set of ip probability spaces follows from (A). It's clear that no information is lost when going from a set of ip probability spaces to a dps. \square

B. T-Algebras and GPS's

Very little in the definition of a dps depends on X being a set of dynamic sets or G_T being a set of ip's. As things often get simpler as they get more abstract, it will be useful to take a step back & generalize the probability theory.

Definition 88. A *t-algebra* is a double, (X, T) , where X is a set of sets and

- 1) For every $x \in X$ there's a $\gamma \in T$ s.t. γ is a partition of x
- 2) For every $t \in T$ there's a $\gamma \in T$ s.t. γ is a partition of some $x \in X$ and $t \subset \gamma$
- 3) If $t_1, t_2 \in T$ then $t_1 - t_2 \in T$

A t-algebra, (X, T) , may be referred to simply by T . As before, $G_T \equiv \{\gamma \in T : \gamma \text{ is a partition of some } x \in X\}$.

Definition 89. A *generalized probability space* (gps) is a triple, (X, T, P) , where (X, T) is a t-algebra and $P : T \rightarrow [0, 1]$ s.t.

- 1) If $t \in T$ is a partition of some $x \in X$ then $P(t) = 1$
- 2) If $t_1, t_2 \in T$ are disjoint and $t_1 \cup t_2 \in T$ then $P(t_1 \cup t_2) = P(t_1) + P(t_2)$

1. $\neg t$ and $[t]$

Definition 90. For t-algebra (X, T) , $t \in T$, $\neg t \equiv \{t' \in T : t' \cap t = \emptyset \text{ and } t' \cup t \in G_T\}$.

For $A \subset T$, $\neg A \equiv \bigcup_{t \in A} \neg t$.

$\neg^{(1)}t \equiv \neg t$, $\neg^{(2)}t \equiv \neg \neg t$, $\neg^{(3)}t \equiv \neg \neg \neg t$, etc.

Theorem 91. For gps (X, T, P) , $t \in T$, $t' \in \neg^{(n)}t$

- 1) If n is odd, $P(t) + P(t') = 1$
- 2) If n is even, $P(t) = P(t')$

Proof. Simple induction over \mathbb{N}^+ . □

Theorem 92. For gps (X, T, P) , $t, t' \in T$, $n, m > 0$

- 1) $t' \in \neg^{(n+m)}t$ iff for some $t'' \in T$, $t'' \in \neg^{(n)}t$ and $t' \in \neg^{(m)}t''$
- 2) $t' \in \neg^{(n)}t$ iff $t \in \neg^{(n)}t'$
- 3) $t \in \neg^{(2n)}t$
- 4) If $m < n$ then $\neg^{(2m)}t \subset \neg^{(2n)}t$ and $\neg^{(2m+1)}t \subset \neg^{(2n+1)}t$

Proof. (1) follows from induction over m . (2) follows from induction over n . (3) follows from induction over n and (1). (4) follows from (1) and (3). □

Definition 93. If (X, T) is a t-algebra and $t \in T$, $[t] \equiv \bigcup_{n \in \mathbb{N}^+} \neg^{(2n)}t$.

Theorem 94. $[t]$ is an equivalence class

Proof. By Thm 92.3, $t \in [t]$

By Thm 92.2, if $t_1 \in [t_2]$ then $t_2 \in [t_1]$

By Thm 92.1, if $t_2 \in [t_1]$ and $t_3 \in [t_2]$ then $t_3 \in [t_1]$ □

Theorem 95. $\neg[t] = \bigcup_{n \in \mathbb{N}^+} \neg^{(2n+1)}t$

Proof. $\neg[t] \equiv \bigcup_{t' \in [t]} \neg t'$. $t'' \in \bigcup_{t' \in [t]} \neg t'$ iff for some $t' \in [t]$, $t'' \in \neg t'$. $t' \in [t]$ iff for some $n \in \mathbb{N}^+$, $t' \in \neg^{(2n)}t$. So $t'' \in \bigcup_{t' \in [t]} \neg t'$ iff for some $n \in \mathbb{N}^+$, $t'' \in \neg^{(2n+1)}t$. □

Theorem 96. If $t' \in \neg[t]$ then $t \in \neg[t']$

Proof. Follows from Thm 95 and Thm 92.2 □

Definition 97. A t-algebra is *simple* if for every $t \in T$, $\neg t = \neg \neg \neg t$.

Theorem 98. If T is a simple t-algebra and $t \in T$ then $[t] = \neg \neg t$ and $\neg[t] = \neg t$

Proof. Clear □

2. $(X_{\mathbb{N}}, T_{\mathbb{N}}, P_{\mathbb{N}})$

In general, dps's will not have simple t-algebras. This can be seen from the fact that, if $t_1 \in \neg t_2$, $t_2 \in \neg t_3$, and $t_4 \in \neg t_3$, $t_1 \cup t_4$ will be a partition of D , but will generally not be an ip.

When t-algebras are not simple, they can get fairly opaque. For example, it's possible to have $t_1 \in [t_2]$, and $t = t_1 \cap t_2$, but $t_1 - t \notin [t_2 - t]$, even though it's clear that for any probability function P , $P(t_1 - t) = P(t_2 - t)$. This could never happen for a simple t-algebra. Fortunately, starting with any t-algebra, it's always possible to use \neg and $[t]$ to build an equivalent simple t-algebra. This will be done iteratively, the first iteration being the t-algebra T_1 :

Definition 99. For t-algebra (X, T) ,

For $t, t' \in T$, $t \perp t'$ if for some $t_1, t_2 \in T$, $t_1 \in \neg[t_2]$, $t \subset t_1$ and $t' \subset t_2$

$$T_1 \equiv \{t \cup t' : t \perp t'\}$$

$$X_1 \equiv \{(\cup t) \cup (\cup t') : t \in T \text{ and } t' \in \neg[t]\}$$

Theorem 100. If (X, T) is a t-algebra then (X_1, T_1) is a t-algebra

Proof. Axioms (1) and (2) hold by the definitions of \perp and X_1 .

For axiom (3), given any $t_1, t_2 \in T_1$, there exist $t'_1, t''_1, t'_2, t''_2 \in T$ s.t. $t'_i \perp t''_i$ and $t_i = t'_i \cup t''_i$. Define $t_3 \equiv (t'_1 - t'_2) - t''_2$ and $t_4 \equiv (t''_1 - t'_2) - t''_2$; $t_3, t_4 \in T$; since $t_3 \subset t'_1$ and $t_4 \subset t''_1$, $t_3 \perp t_4$, so $t_3 \cup t_4 \in T_1$. $t_1 - t_2 = t_3 \cup t_4$ so $t_1 - t_2 \in T_1$. \square

We now turn to constructing a probability function for T_1 .

Theorem 101. If (X, T, P) is a gps, $t_1 \perp t_2$, $t_3 \perp t_4$, and $t_1 \cup t_2 = t_3 \cup t_4$, then $P(t_1) + P(t_2) = P(t_3) + P(t_4) \in [0, 1]$; further, if $t_1 \in \neg[t_2]$ then $P(t_1) + P(t_2) = 1$

Proof. Take $t'_1 = t_1 \cap t_3$, $t'_2 = t_1 \cap t_4$, $t'_3 = t_2 \cap t_3$, $t'_4 = t_2 \cap t_4$. All t'_i are disjoint, and each t_i is a union of two of the t'_j , so $P(t_1) + P(t_2) = P(t'_1) + P(t'_2) + P(t'_3) + P(t'_4) = P(t_3) + P(t_4)$.

If $t_1 \in \neg[t_2]$ then $P(t_1) + P(t_2) = 1$ by Thm 91.1.

More generally, if $t_1 \perp t_2$, for some $t_5, t_6 \in T$, $t_6 \in \neg[t_5]$, $t_1 \subset t_5$, and $t_2 \subset t_6$. $P(t_5) + P(t_6) = 1$, so $P(t_1) + P(t_2) \in [0, 1]$. \square

Definition 102. If (X, T, P) is a gps, $P_1 : T_1 \rightarrow [0, 1]$ s.t. for $t_1, t_2 \in T$, $t_1 \perp t_2$, $P_1(t_1 \cup t_2) = P(t_1) + P(t_2)$.

Theorem 103. *If (X, T, P) is a gps then (X_1, T_1, P_1) is a gps.*

Proof. Follows from Thm 100 and Thm 101. □

This process can now be repeated; starting with (X_1, T_1, P_1) , (X_2, T_2, P_2) can be constructed as $((X_1)_1, (T_1)_1, (P_1)_1)$.

Definition 104. If (X, T, P) is a gps, $(X_0, T_0, P_0) \equiv (X, T, P)$ and $(X_{n+1}, T_{n+1}, P_{n+1}) \equiv ((X_n)_1, (T_n)_1, (P_n)_1)$.

Theorem 105. *If (X, T, P) is a gps*

- 1) (X_n, T_n, P_n) is a gps
- 2) $t \in T_n$ iff for some finite, pairwise disjoint $A \subset T$ with not more than 2^n members s.t. $\bigcup A \in G_{T_n}$, some $B \subset A$, $t = \bigcup B$
- 3) If $t \in T_n$ and $A \subset T$ is finite, pairwise disjoint, and $t = \bigcup A$ then $P_n(t) = \sum_{x \in A} P(x)$
- 4) If $t \in T_n$ and $t \in T_m$ then $P_n(t) = P_m(t)$

Proof. For 1-3, Straightforward induction. (4) follows from (2) & (3). □

Definition 106. If (X, T) is a t-algebra

$$X_{\mathbb{N}} \equiv \bigcup_{n \in \mathbb{N}^+} X_n$$

$$T_{\mathbb{N}} \equiv \bigcup_{n \in \mathbb{N}^+} T_n$$

If (X, T, P) is a dps, $P_{\mathbb{N}} : T_{\mathbb{N}} \rightarrow [0, 1]$ s.t. if $t \in T_n$ then $P_{\mathbb{N}}(t) = P_n(t)$

Theorem 107. *If (X, T, P) is a gps then $(X_{\mathbb{N}}, T_{\mathbb{N}}, P_{\mathbb{N}})$ is a simple gps*

Proof. That $(X_{\mathbb{N}}, T_{\mathbb{N}})$ is a t-algebra follows from the fact that if $t \in T_{\mathbb{N}}$ then for some $m \in \mathbb{N}^+$, all $n > m$, t is an element of T_n , and all T_n are t-algebras. That $(X_{\mathbb{N}}, T_{\mathbb{N}}, P_{\mathbb{N}})$ is a gps follows similarly.

It remains to show that $T_{\mathbb{N}}$ is simple. Take “ \neg_n ” to be “ \neg ” defined on T_n and “ $\neg_{\mathbb{N}}$ ” to be “ \neg ” defined on $T_{\mathbb{N}}$. For $t \in T_{\mathbb{N}}$, if $t' \in \neg_{\mathbb{N}} \neg_{\mathbb{N}} \neg_{\mathbb{N}} t$ then for some $t_1, t_2 \in T_{\mathbb{N}}$, $t' \in \neg_{\mathbb{N}} t_2$, $t_2 \in \neg_{\mathbb{N}} t_1$, and $t_1 \in \neg_{\mathbb{N}} t$, so for some $m, n, p \in \mathbb{N}$, $t' \in \neg_m t_2$, $t_2 \in \neg_n t_1$, and $t_1 \in \neg_n t$, in which case, with $q = \text{lub}(\{m, n, p\})$, $t' \in \neg_{q+1} t$, and so $t' \in \neg_{\mathbb{N}} t$ □

The following will be of occasional use.

Definition 108. $[t]_n$ and $\neg_n t$ may be use to refer to $[t]$ and $\neg t$ on (X_n, T_n) . $[t]_{\mathbb{N}}$ and $\neg_{\mathbb{N}} t$ may be use to refer to $[t]$ and $\neg t$ on $(X_{\mathbb{N}}, T_{\mathbb{N}})$.

Theorem 109. *If (X, T, P) is a gps*

1) *$t \in T_{\mathbb{N}}$ iff for some finite, pairwise disjoint $A \subset T$ s.t. $\bigcup A \in G_{T_{\mathbb{N}}}$, some $B \subset A$, $t = \bigcup B$*

2) *If $A \subset T$ is finite, pairwise disjoint, and $\bigcup A \in T_{\mathbb{N}}$ then $P_{\mathbb{N}}(\bigcup A) = \sum_{t \in A} P(t)$*

Proof. Follows from Thm 105, □

3. Convergence on a GPS

Earlier, the question of countable convergence was deferred. Some basic concepts will now be presented.

Definition 110. For t-algebra (X, T) , \mathcal{A} is a *c-set* if it is a countable partition of some $y \in X$, and there exists a sequence on \mathcal{A} , $(A_n)_{n \in \mathbb{N}}$, $(\mathcal{A} = \{A_n : n \in \mathbb{N}\})$ s.t. for all $n \in \mathbb{N}$, $\bigcup_{i \leq n} A_i \in T$.

Theorem 111. *If (X, T) is a t-algebra, \mathcal{A} is a c-set iff it is a countable partition of some $y \in X$ and for any finite $B \subset \mathcal{A}$, $\bigcup B \in T$*

Proof. \Rightarrow There exists a sequence $(A_n)_{n \in \mathbb{N}}$ s.t. $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ and for all $n \in \mathbb{N}$, $\bigcup_{i \leq n} A_i \in T$.

For all $A_i \in \mathcal{A}$, $A_i = \bigcup_{j \leq i} A_j - \bigcup_{k \leq i-1} A_k$, so $A_i \in T$.

For each $b \in B$ there exists a unique $i \in \mathbb{N}$ s.t. $A_i = b$; take k to be the largest such number. $\bigcup B \subset \bigcup_{i \leq k} A_i \in T$. Since (X, T) is a t-algebra, B is a finite subset of T , and $\bigcup B$ is a subset of an element of T , $\bigcup B \in T$.

\Leftarrow Immediate. □

If (X, T) is a t-algebra and $(t_n)_{n \in \mathbb{N}}$ is a sequence of elements of T s.t. for all $i \in \mathbb{N}$, $t_i \subset t_{i+1}$ and $\bigcup_{i \in \mathbb{N}} t_i$ is the partition of some $S \in X$, then $\mathcal{A} \equiv \{t_{i+1} - t_i : i \in \mathbb{N}\}$ is a c-set. Further, if \mathcal{A} is a c-set and $(A_n)_{n \in \mathbb{N}}$ is any sequence of it's elements then $t_i = \bigcup_{j \leq i} A_j$ form such a sequence of elements of T , so these two notions are equivalent.

Theorem 112. *If (X, T, P) is a gps and \mathcal{A} is a c-set then $\sum_{x \in \mathcal{A}} P(x) \leq 1$.*

Proof. If $(A_n)_{n \in \mathbb{N}}$ is any sequence on \mathcal{A} , for all $N \in \mathbb{N}$, $\sum_{n \leq N} P(A_n) = P(\bigcup_{n \leq N} A_n) \leq 1$. □

Definition 113. A gps, (X, T, P) , is *convergent* if for every c-set, \mathcal{A} , $\sum_{x \in \mathcal{A}} P(x) = 1$.

Theorem 114. If (X, T, P) is a convergent dps, \mathcal{A} and \mathcal{B} are c-sets, $Z \subset \mathcal{A}$, $Y \subset \mathcal{B}$, and $\bigcup Z = \bigcup Y$ then $\sum_{x \in Z} P(x) = \sum_{x \in Y} P(x)$.

Proof. Take $C \equiv \{\alpha : \text{for some } a \in Y, b \in Z, \alpha = a \cap b \text{ and } \alpha \neq \emptyset\}$ and $V \equiv (\mathcal{B} - Y) \cup C$. V has the following properties: every element of V is an element of T ; for every finite $\alpha \subset V$ there exists a finite $b \subset B$ s.t. $\bigcup \alpha \subset \bigcup b$, and so $\bigcup \alpha \in T$; and finally V is pairwise disjoint & $\bigcup V = \bigcup B$. Therefore V is a c-set.

$\sum_{x \in B - Y} P(x) + \sum_{x \in C} P(x) = \sum_{x \in V} P(x) = 1 = \sum_{x \in B} P(x) = \sum_{x \in B - Y} P(x) + \sum_{x \in Y} P(x)$, so $\sum_{x \in C} P(x) = \sum_{x \in Y} P(x)$. Proceeding similarly with $(A - Z) \cup C$ yields $\sum_{x \in C} P(x) = \sum_{x \in Z} P(x)$. Therefore $\sum_{x \in Z} P(x) = \sum_{x \in Y} P(x)$. \square

Definition 115. If (X, T) is a t-algebra, $T_c \equiv \{x : \text{For some c-set, } A, \text{ some } Z \subset A, x = \bigcup Z\}$.

If (X, T, P) is a convergent dps, $P_c : T_c \rightarrow [0, 1]$ s.t. if A is a c-set and $B \subset A$ then $P_c(\bigcup B) = \sum_{x \in B} P(x)$.

Theorem 116. If (X, T, P) is convergent $A \subset T_c$ is countable, pairwise disjoint, and $\bigcup A \in T_c$ then $\sum_{x \in A} P_c(x) = P_c(\bigcup A)$.

Proof. For every $x \in A$ take Y_x to be a c-set s.t. for $B_x \subset Y_x$, $\bigcup B_x = x$. Take Z to be a c-set s.t. for $D \subset Z$, $\bigcup D = \bigcup A$. Finally, for each $x \in A$, take $C_x \equiv \{\alpha : \text{for some } a \in B_x, b \in D, \alpha = a \cap b \text{ \& } \alpha \neq \emptyset\}$. Note that $\bigcup C_x = x$.

Each $(Y_x - B_x) \cup C_x$ is a c-set, so $\sum_{t \in C_x} P_c(t) = P_c(x)$. With $C \equiv \bigcup_{x \in A} C_x$, $(Z - D) \cup C$ is also a c-set, so $P_c(\bigcup A) = \sum_{x \in A} \sum_{t \in C_x} P_c(t) = \sum_{x \in A} P_c(x)$. \square

If (X, T) is a t-algebra, there's no guarantee that (X, T_c) will be a t-algebra; in particular, if A and B are c-sets, $(\bigcup A) \cap (\bigcup B)$ might not be an element of T_c . [5] Also, if (X, T, P) is convergent, $(X_{\mathbb{N}}, T_{\mathbb{N}}, P_{\mathbb{N}})$ may not be (though if $(X_{\mathbb{N}}, T_{\mathbb{N}}, P_{\mathbb{N}})$ is convergent then (X, T, P) is). Because convergence on $(X_{\mathbb{N}}, T_{\mathbb{N}}, P_{\mathbb{N}})$ is a useful quality for a gps to possess, the following simplified notation will be used.

Definition 117. If (X, T) is a t-algebra, $T_{\Upsilon} \equiv (T_{\mathbb{N}})_c$. If (X, T, P) is a dps, it is Υ -convergent if $(X_{\mathbb{N}}, T_{\mathbb{N}}, P_{\mathbb{N}})$ is convergent. If (X, T, P) is an Υ -convergent dps then $P_{\Upsilon} \equiv (P_{\mathbb{N}})_c$.

The easiest way to remember this is that Υ has absolutely nothing to do with these properties.

4. T_S , $T_{S\mathbb{N}}$, and $T_{\mathbb{N}}^S$

Definition 118. If (X, T) is a t-algebra and $S \in X$ then

$$T_S \equiv \{t \in T : \text{for some } \gamma \in G_T, \bigcup \gamma = S \text{ and } t \subset \gamma\}$$

$$T_{S\mathbb{N}} \equiv (T_S)_{\mathbb{N}} \text{ and } T_{\mathbb{N}}^S \equiv (T_{\mathbb{N}})_S.$$

It follows readily that $(\{S\}, T_S)$ is a t-algebra, both $(\{S\}, T_{S\mathbb{N}})$ and $(\{S\}, T_{\mathbb{N}}^S)$ are simple t-algebras, and $T_{S\mathbb{N}} \subset T_{\mathbb{N}}^S$. One question that arises is, under what conditions will $T_{S\mathbb{N}} = T_{\mathbb{N}}^S$; that is, when does the rest of T yield no further information about the probabilities on S ? There are a number of ways in which this can occur; the most obvious is: for all $S' \in X$, $S' \neq S$, $S \cap S' = \emptyset$ (or more generally, for all $\gamma \in G_{T_S}$, $\gamma' \in G_{T_{S'}}$, if $S \neq S'$ then $\gamma \cap \gamma' = \emptyset$). In such cases, T_S is completely independent of the rest of T . Another way in which $T_{S\mathbb{N}} = T_{\mathbb{N}}^S$ is if the structure of T as a whole can be mapped onto T_S ; this possibility will now be sketched.

(In the following definition, $\mathbf{P}(S)$ is the power set of S)

Definition 119. If (X, T) and $(\{S\}, T')$ are t-algebras, $f : \bigcup T \rightarrow \mathbf{P}(\bigcup T')$ reduces T to T' if

- 1) If $\gamma \in G_T$ then $\bigcup f[\gamma] \in G_{T'}$
- 2) If $\gamma \in G_T$, $\alpha, \beta \in \gamma$, and $\alpha \neq \beta$ then $f(\alpha) \cap f(\beta) = \emptyset$
- 3) If $\alpha \in \bigcup T$ and $\alpha \subset S$ then $f(\alpha) = \{\alpha\}$

Theorem 120. If f reduces T to T' then it reduces T_1 to T'_1

Proof. For $t_1, t_2 \in T$, assume $t_1 \in \neg t_2$; by (2), $f[t_1] \cap f[t_2] = \emptyset$ and by (1) $\bigcup f[t_1 \cup t_2] \in G_{T'}$, so $\bigcup f[t_1] \in \neg \bigcup f[t_2]$. It follows from the definition of $\neg^{(n)}$ that if $t_1 \in \neg^{(n)} t_2$ then $\bigcup f[t_1] \in \neg^{(n)} \bigcup f[t_2]$. (1) & (2) in the definition of reduction follow immediately from this. (3) continues to hold because $\bigcup T_1 = \bigcup T$. \square

Theorem 121. If f reduces T to T' then it reduces $T_{\mathbb{N}}$ to $T'_{\mathbb{N}}$

Proof. By Thm 120, f reduces T_n to T'_n .

If $\gamma \in G_{T_{\mathbb{N}}}$ then for some n , $\gamma \in G_{T_n}$, and so (1) and (2) hold. (3) hold because $\bigcup T_{\mathbb{N}} = \bigcup T$. \square

Theorem 122. If (X, T) is a t-algebra, $S \in X$, and there exists an f that reduces T to T_S then $T_{\mathbb{N}}^S = T_{S\mathbb{N}}$.

Proof. In all cases, $T_{S\mathbb{N}} \subset T_{\mathbb{N}}^S$

Take any $t \in T_{\mathbb{N}}^S$ and any $\gamma \in G_{T\mathbb{N}}$ s.t. $t \subset \gamma$. By Thm 121 f reduces $T_{\mathbb{N}}$ to $T_{S\mathbb{N}}$, so by (1) and (3) in the definition of a reduction, $t \subset \bigcup f[\gamma] \in G_{T_{S\mathbb{N}}}$; therefore $t \in T_{S\mathbb{N}}$ and so $T_{\mathbb{N}}^S \subset T_{S\mathbb{N}}$. \square

C. Nearly Compatible Sets

Now to return to dps's and, in particular, the $(X_{\mathbb{N}}, T_{\mathbb{N}}, P_{\mathbb{N}})$ construction for dps's. Because the only additional requirement imposed on dps's is that all $\gamma \in G_T$ must be ip's, and because this only places an indirect restriction on the makeup of “ X ” in dps (X, T, P) , dps's of the form $(\{S\}, T, P)$ will be of greatest interest. Naturally, any dps may be decomposed into a set such dps's, one for each $S \in X$, and the results of this section may be applied to those component parts.

Definition 123. A set of companionable sets, A , is *nearly compatible* if it is pairwise disjoint and for every $\alpha, \beta \in A$, every $(\lambda, p) \in \text{Uni}(\bigcup A)$ either $\alpha_{\rightarrow(\lambda, p)} = \beta_{\rightarrow(\lambda, p)}$ or $\alpha_{\rightarrow(\lambda, p)} \cap \beta_{\rightarrow(\lambda, p)} = \emptyset$.

All compatible sets are nearly compatible, but the converse does not hold. The basic result of this section is that for a dps, $(\{S\}, T, P)$, while elements of $G_{T\mathbb{N}}$ need not be compatible, they must be nearly compatible.

Definition 124. \mathcal{A} is a *nearly compatible collection* (ncc) if it is pairwise disjoint and $\bigcup \mathcal{A}$ is nearly compatible. \mathcal{A} is an *S-nccp* if it is an ncc and $\bigcup \mathcal{A}$ is a partition of S .

Theorem 125. If S is a dynamic set and $\{t_1, t_2\}$, $\{t_2, t_3\}$, and $\{t_3, t_4\}$ are *S-nccp's*, then so is $\{t_1, t_4\}$.

Proof. Take $\alpha \in t_1$, $\beta \in t_4$ and assume $\alpha_{\rightarrow(\lambda, p)} \cap \beta_{\rightarrow(\lambda, p)} \neq \emptyset$. With $\bar{s}[-\infty, \lambda] \in \alpha_{\rightarrow(\lambda, p)} \cap \beta_{\rightarrow(\lambda, p)}$, there must be a $\eta \in t_2$ and a $\nu \in t_3$ s.t. $\bar{s}[-\infty, \lambda] \in \eta_{\rightarrow(\lambda, p)}$ and $\bar{s}[-\infty, \lambda] \in \nu_{\rightarrow(\lambda, p)}$. Since $t_1 \cup t_2$, $t_2 \cup t_3$, and $t_3 \cup t_4$ are nearly compatible, $\alpha_{\rightarrow(\lambda, p)} = \eta_{\rightarrow(\lambda, p)} = \nu_{\rightarrow(\lambda, p)} = \beta_{\rightarrow(\lambda, p)}$. \square

Thm 125 is the key property of nearly compatible sets. Essentially, if Y is an attribute s.t. all ip's are Y , and Y is transitive in the sense of Thm 125, then all $\gamma \in G_{T\mathbb{N}}$ are Y . Near compatibility is a close approximation of compatibility that possesses this property.

Theorem 126. *If $(\{S\}, T, P)$ is a dps, $t_1 \in T$, and $t_2 \in \neg[t_1]$ then $\{t_1, t_2\}$ is an S -nccp.*

Proof. If $t_1 \in \neg t_2$ then t_1 and t_2 are disjoint and $t_1 \cup t_2$ is a compatible (and so nearly compatible) partition, so $\{t_1, t_2\}$ is an T -nccp. The result is now immediate from Thm 125. \square

Note that if $t_1 \in \neg[t_2]$ there are no grounds to conclude that $t_1 \cup t_2$ is compatible, and in general it won't be.

Theorem 127. *If $(\{S\}, T, P)$ is a dps, $x_1 \in T_n$, and $x_2 \in \neg_n[x_1]_n$ then $\{x_1, x_2\}$ is a S -nccp.*

Proof. Holds for $n = 0$ by Thm 126. Assume it holds for $n = m$. For $n = m + 1$, if $x_2 \in \neg_{m+1}x_1$ then $x_1 \cup x_2 \in G_{T(m+1)}$; therefore there must be a $y_1, y_2 \in T_m$ s.t. $y_2 \in \neg_m[y_1]_m$ and $y_1 \cup y_2 = x_1 \cup x_2$. By assumption on $n = m$, $x_1 \cup x_2 = y_1 \cup y_2$ is nearly compatible, so $\{x_1, x_2\}$ is a S -nccp. From Thm 125 it then follows that if $x_2 \in \neg_{m+1}[x_1]_{m+1}$ then $\{x_1, x_2\}$ is an S -nccp. \square

Theorem 128. *If $(\{S\}, T, P)$ is a dps then*

- 1) *If $\gamma \in G_{T\mathbb{N}}$ then for some finite S -nccp $A \subset T$, $\gamma = \bigcup A$*
- 2) *If $t \in T_{\mathbb{N}}$ then for some finite ncc, $A \subset T$, $t = \bigcup A$*

Proof. Immediate from Thm 127. \square

The richer a dps's t-algebra is, the more information the dps carries. A t-algebra contains minimal information if $G_{T\mathbb{N}}$ contains only the original ip's; in that case, no information can be derived from the dps as a whole that isn't known by considering the individual $(\gamma, T_\gamma, P_\gamma)$ in isolation. As the t-algebra grows richer, more information can be derived from it. Thm 128 indicates the outer limit on the information that can be derived from a dps on a single dynamic set.

Definition 129. If $(\{S\}, T, P)$ is a dps, it is *maximal* if $\gamma \in G_{T\mathbb{N}}$ iff there's a finite S -nccp, $A \subset T$, s.t. $\bigcup A = \gamma$.

If $(\{S\}, T, P)$ and $(\{S\}, T', P')$ are dps's, $T \subset T'$, and $P = P'|_T$, then if T is maximal, $(\{S\}, T', P')$ will give no further information about $(\{S\}, T, P)$. However, if $(\{S\}, T, P)$ is not maximal, but $(\{S\}, T', P')$ is, this immediately yields a wealth of information about

$(\{S\}, T, P)$, indeed more information than $(\{S\}, T_{\mathbb{N}}, P_{\mathbb{N}})$ yields; it means that the following construct forms a simple gps:

$$\begin{aligned} G_{T_e} &\equiv \{\gamma : \text{for some finite } S\text{-nccp, } A \subset T, \gamma \equiv \bigcup A\} \\ T_e &\equiv \{t : \text{for some finite ncc, } A \subset T, \text{ some } \gamma \in G_{T_e}, t = \bigcup A \subset \gamma\} \\ P_e : T_e &\rightarrow [0, 1] \text{ s.t. if } A \subset T \text{ is a finite ncc and } \bigcup A \equiv t \in T_e \text{ then } P_e(t) = P'_{\mathbb{N}}(t) = \\ &\sum_{t' \in A} P(t'). \end{aligned}$$

Note that $T_e = T_{\mathbb{N}}$ only if $(\{S\}, T, P)$ is maximal. In all other cases, $(\{S\}, T_e, P_e)$ contains more information than $(\{S\}, T_{\mathbb{N}}, P_{\mathbb{N}})$.

For Υ -convergent dps's, this can be pushed further.

Definition 130. If $(\{S\}, T, P)$ is a dps, it is ω -maximal if for every countable S -nccp, $A \subset T$, every finite $B \subset A$, $\bigcup B \in T_{\mathbb{N}}$.

$(\{S\}, T, P)$ is Υ -maximal if it is ω -maximal and Υ -convergent.

For an ω -maximal dps, every countable S -nccp composed of elements of T is a c-set in $T_{\mathbb{N}}$. This means that for a Υ -maximal dps, $\gamma \in G_{T\Upsilon}$ iff there's a countable S -nccp, $A \subset T$, s.t. $\bigcup A = \gamma$. It was mentioned previously that, in general, $(\{S\}, T_{\Upsilon}, P_{\Upsilon})$ is not a gps; however it can be seen that for Υ -maximal dps's, $(\{S\}, T_{\Upsilon}, P_{\Upsilon})$ is not only a gps, it's a simple gps.

Returning to the case where $(\{S\}, T, P)$ and $(\{S\}, T', P')$ are dps's, $T \subset T'$, and $P = P'|_T$, if $(\{S\}, T', P')$ is Υ -maximal then the following construct forms a simple gps:

$$\begin{aligned} G_{T_{\varepsilon}} &\equiv \{\gamma : \text{for some countable } S\text{-nccp, } A \subset T, \gamma \equiv \bigcup A\} \\ T_{\varepsilon} &\equiv \{t : \text{for some countable ncc, } A \subset T, \text{ some } \gamma \in G_{T_{\varepsilon}}, t = \bigcup A \subset \gamma\} \\ P_{\varepsilon} : T_{\varepsilon} &\rightarrow [0, 1] \text{ s.t. if } A \text{ is a countable ncc and } \bigcup A \equiv t \in T_{\varepsilon} \text{ then } P_{\varepsilon}(t) = P'_{\Upsilon}(t) = \\ &\sum_{t' \in A} P(t'). \end{aligned}$$

When it's applicable, this is, of course, a very useful construct.

D. Deterministic & Herodotistic Spaces

A system is deterministic if a complete knowledge of the present yields a complete knowledge of the future; it is herodotistic if complete knowledge of the present yields a complete knowledge of the past. More formally:

Definition 131. A dynamic set, S , is *deterministic* if for every $(\lambda, p) \in \text{Uni}(S)$, $S_{(\lambda, p) \rightarrow}$ is a singleton (that is, $S_{(\lambda, p) \rightarrow}$ has only one element).

S is *herodotistic* if for every $(\lambda, p) \in \text{Uni}(S)$, $S_{\rightarrow(\lambda, p)}$ is a singleton.

In this section deterministic and herodotistic dps's will be investigated, as will deterministic/herodotistic “universes”, where the e-automata as whole is deterministic/herodotistic though the system being measured might not be. It will follow from the various results that for the probabilities seen in quantum physics to hold, neither the systems being investigated nor the universe as a whole can be either deterministic or herodotistic.

No other material in this article will depend on the material in this section.

1. DPS's on Deterministic & Herodotistic Spaces

Theorem 132. *If a dynamic set, S , is deterministic or herodotistic then S is a dynamic space all its subsets are subspaces.*

Proof. For $A \subset S$, $\bar{p}_1, \bar{p}_2 \in A$, $\bar{p}_1(\lambda) = \bar{p}_2(\lambda)$, if S is herodotistic then $\bar{p}_1[-\infty, \lambda] = \bar{p}_2[-\infty, \lambda]$ so $\bar{p}_1[-\infty, \lambda] \circ \bar{p}_2[\lambda, \infty] = \bar{p}_2 \in A$. Similarly, if S is deterministic then $\bar{p}_1[-\infty, \lambda] \circ \bar{p}_2[\lambda, \infty] = \bar{p}_1 \in A$ □

Theorem 133. 1) *If D is deterministic then for any $X \subset D$, $\lambda \in \Lambda_D$, $X = +X[-\infty, \lambda]$*
 2) *If D is herodotistic then for any $X \subset D$, $\lambda \in \Lambda_D$, $X = +X[\lambda, \infty]$*

Proof. 1) For any $\bar{p}[-\infty, \lambda] \in X[-\infty, \lambda]$ there exists only one $\bar{p}' \in D$ s.t. $\bar{p}'[-\infty, \lambda] = \bar{p}[-\infty, \lambda]$.

2) Similar □

Theorem 134. 1) *If D is deterministic then $A, B \subset D$ are compatible iff they are disjoint and bounded from below*

2) *If D is herodotistic then $A, B \subset D$ are compatible iff they are disjoint and bounded from above*

Proof. 1) If A and B are disjoint then $A[-\infty, \lambda] \cap B[-\infty, \lambda] = \emptyset$ for all λ .

2) If $p \in A(\lambda) \cap B(\lambda)$ then since $\rightarrow(\lambda, p)$ has only one element, $A_{\rightarrow(\lambda, p)} = B_{\rightarrow(\lambda, p)}$. □

Definition 135. If $(\{S\}, T, P)$ is a dps, it is *closed under combination* if for all pairwise disjoint $A \subset T$ s.t. $\bigcup A$ is an ip, $\bigcup A \in G_T$.

Theorem 136. *If (X, T, P) is a dps and for all $S \in X$, S is either deterministic or herodotistic and $(\{S\}, T_S, P_s)$ is closed under combination, then for all $t_1, t_2 \in T$ s.t. $\bigcup t_1 = \bigcup t_2$*

1) *If $\gamma \in G_T$ and $t_1 \subset \gamma$ then $(\gamma - t_1) \bigcup t_2 \in G_T$*

2) *$P(t_1) = P(t_2)$*

Proof. (1) follows from Thm 134 and (2) follows from (1). \square

Note the importance of closure under combination. One can map any dps, (X, T, P) , it onto a herodotistic dps as follows. For every $S \in X$, $\bar{p} \in S$, define \bar{p}_h by $\bar{p}_h(\lambda) = (\bar{p}(\lambda), \bar{p}[0, \lambda])$. With $S_h \equiv \{\bar{p}_h : \bar{p} \in S\}$, S_h is herodotistic. One can then readily construct (X_h, T_h, P_h) , which will be a dps, but in most cases will not be closed under combination; indeed, since $\bigcup t_h = \bigcup t'_h$ iff $\bigcup t = \bigcup t'$ and $P(t) = P_h(t_h)$, the conclusion of above theorem will generally not apply to (X_h, T_h, P_h) .

2. DPS's in Deterministic Universes

Theorem 137. *If partition γ is decided by an idea e-automata, $Z = (D, I, F)$ s.t. D is deterministic, then for some $\lambda_0 \in \Lambda$, all $\alpha \in \gamma$, α is $sbb\lambda_0$ and $wba\lambda_0$.*

Proof. A: For every $(s, e), (s, e') \in I$, $\Sigma_{(s,e) \rightarrow} = \Sigma_{(s,e') \rightarrow}$

- With $\bar{s}[0, 0]$ s.t. $\bar{s}[0, 0](0) = s$, $\bar{s}[0, 0] \in \Sigma_{(s,e)} \cap \Sigma_{(s,e')}$ (since $(s, e), (s, e') \in I$), so the result follows from Thm 39. -

B: Taking $S \equiv \bigcup \gamma$, for any $s \in S(0)$, $(\mathcal{O}_F)_{(0,s) \rightarrow}$ is either a singleton or empty

- Follows from definition of “deterministic” and (A) -

C: For some $\lambda_0 \in \Lambda$, all $\alpha \in \gamma$, there's a $\lambda_\alpha \in \Lambda$, s.t. α is $sbb\lambda_0$, $wba\lambda_\alpha$, and for all $s \in \alpha(\lambda_0)$, $\alpha_{s \rightarrow}[\lambda_0, \lambda_\alpha]$ is a singleton

- Follows from (B) and the definition of being decided by an e-automata -

D: For all $\alpha, \beta \in \gamma$, if $\alpha \neq \beta$ then $\alpha(\lambda_0) \cap \beta(\lambda_0) = \emptyset$

- Follows from (B) and the definition of being decided by an e-automata -

The theorem is immediate from (C) & (D). \square

Definition 138. If Y is a collection subsets of dynamic set S then Y is *cross-section (at λ)* if it is pairwise disjoint and for all $y \in Y$, y is a subspace, $sbb\lambda$, and $wba\lambda$.

Theorem 139. 1) *If Y is a cross-section then it's compatible.*

2) *If Y is cross-section at λ then $\{\bigcup Y\}$ is a cross-section at λ*

Proof. Clear □

Definition 140. A t -algebra is *deterministically decidable* if for all $t \in T$, t is a cross-section.

It is *deterministically normal* if it is deterministically decidable and

1) If $t \in T$ then $\{\bigcup t\} \in T$

2) If $A \subset T$ is finite and $\bigcup A$ is a partition & a cross-section then $\bigcup A \in G_T$.

(X, T, P) is a *deterministically decidable dps* if (X, T) is deterministically decidable & *deterministically normal* if (X, T) is deterministically normal.

Theorem 141. *If (X, T, P) is deterministically normal dps*

1) *For any $t, t' \in T$, if $\{\bigcup t, \bigcup t'\}$ is a cross-section and a partition of some $S \in X$ then $t' \in \neg[t]$.*

2) *For all $t_1, t_2 \in T$ s.t. $\bigcup t_1 = \bigcup t_2$, $P(t_1) = P(t_2)$*

Proof. 1) For any $\gamma \in G_T$ s.t. $t \subset \gamma$, $t \cup \{\bigcup(\gamma - t)\}$ is a cross-section. Since $\bigcup t' = \bigcup(\gamma - t)$, $t \cup \{\bigcup t'\}$ is a cross-section. Therefore $t \cup \{\bigcup t'\}$, $\{\bigcup t\} \cup t'$, and $\{\bigcup t\} \cup \{\bigcup t'\}$ are all elements of G_T .

2) Take any $\gamma \in G_T$ s.t. $t_1 \subset \gamma$. It follows from (1) that $\gamma - t_1 \in \neg[t_2]$, so $t_1 \in [t_2]$ □

If $X = \{S\}$ then (1) becomes - For any $t, t' \in T$, $t' \in \neg[t]$ iff $\{\bigcup t, \bigcup t'\}$ is a cross-section and a partition of S .

3. DPS's in Herodotistic Universes

Definition 142. If t is a compatible set, it is *herodotistically decidable* if for some $\lambda_0 \in \Lambda$, all $\alpha \in t$, there's a $\lambda_\alpha \in \Lambda$ s.t. $\alpha = +\alpha[\lambda_0, \lambda_\alpha]$, and for all $s \in \alpha(\lambda_\alpha)$, $\alpha \rightarrow_s[\lambda_0, \lambda_\alpha]$ is a singleton.

Under these conditions, t is said to be *born at* λ_0

A t -algebra (X, T) is *herodotistically decidable* if every $t \in T$ is herodotistically decidable (which holds iff every $\gamma \in G_T$ is herodotistically decidable).

Theorem 143. *If γ is decided by an ideal e -automata, (D, I, F) , and D is herodotistic, then γ is herodotistically decidable.*

Proof. If D is herodotistic then for any $e \in F$, any $(s, e') \in \theta_e(\Lambda(\mathcal{O}_e))$, $\Sigma_{(s, e')}$ is a singleton, and by Thm 55.3 for any $(s, e'), (s, e'') \in \theta_e(\Lambda(\mathcal{O}_e))$, $\Sigma_{(s, e')} = \Sigma_{(s, e'')}$. The theorem then follows immediately from the definition of being decided by an ideal e -automata. □

Theorem 144. *If $(\cup t_1) \cap (\cup t_2) = \emptyset$, both t_1 and t_2 are herodotistically decidable, and there exists a λ s.t. both t_1 and t_2 are born at λ , then t_1 and t_2 are nearly compatible*

Proof. Immediate from the definitions of herodotistically decidable and nearly compatible. \square

Let's say that for herodotistically decidable t-algebra, (X, T) , $t_1, t_2 \in T$ are *sympathetic* if $\cup t_1 = \cup t_2$ and there exists a $t' \in \neg t_1$, $\lambda \in \Lambda$ s.t. both t_2 and t' are born at λ . Thm 144 means that for a maximal dps (or a dps that can be embedded in a maximal dps), if t_1 and t_2 are sympathetic then $P(t_1) = P(t_2)$. It will soon be seen that this is incompatible with quantum probabilities.

Note that being herodotistic has a less dramatic effect on an ideal e-automata than being deterministic. This is because, for ideal e-automata, the environment is assumed to remember something about the past (due being all-reet), but know little about the future (due to being unbiased).

V. APPLICATION TO QUANTUM MEASUREMENT

A. Preliminary Matters

In this part, results from the prior sections will be applied to quantum physics as described by the Hilbert Space formalism. In order to do this, a few preliminary matters need to be addressed.

1. A Note On Paths

Quantum physics is often viewed in terms of transitions between states, rather than in terms of system paths; an obvious exception being the path integration formalism. Before proceeding, it may be helpful to start by describing conditions under which state transitions can be re-represented as paths.

Start by taking $(\lambda_1, s_1) \Rightarrow (\lambda_2, s_2)$ to mean that state s_1 at time λ_1 can transition to state s_2 at time λ_2 . Paths may then be defined as any parametrized function, \bar{s} , s.t. for all $\lambda_1 < \lambda_2$, $(\lambda_1, \bar{s}(\lambda_1)) \Rightarrow (\lambda_2, \bar{s}(\lambda_2))$. To show that this set of paths is equivalent to the

transition relation, it must be shown that if $(\lambda_1, s_1) \Rightarrow (\lambda_2, s_2)$ then there exists a path, \bar{s} , s.t. $\bar{s}(\lambda_1) = s_1$ and $\bar{s}(\lambda_2) = s_2$ (the converse clearly holds).

In quantum mechanics, the \Rightarrow relation has the following properties:

- 1) If $(\lambda_1, s_1) \Rightarrow (\lambda_2, s_2)$ then $\lambda_1 < \lambda_2$
- 2) For all (λ, s) , all $\lambda' < \lambda$, there's a s' s.t. $(\lambda', s') \Rightarrow (\lambda, s)$
- 3) For all (λ, s) , all $\lambda' > \lambda$, there's a s' s.t. $(\lambda, s) \Rightarrow (\lambda', s')$
- 4) For all $(\lambda_1, s_1), (\lambda_2, s_2)$ s.t. $(\lambda_1, s_1) \Rightarrow (\lambda_2, s_2)$, all $\lambda_1 < \lambda' < \lambda_2$, there's a s' s.t. $(\lambda_1, s_1) \Rightarrow (\lambda', s')$ and $(\lambda', s') \Rightarrow (\lambda_2, s_2)$.

If the measurements are strongly unbiased, then the following holds:

- 5) If $(\lambda_1, s_1) \Rightarrow (\lambda_2, s_2)$, and $(\lambda_2, s_2) \Rightarrow (\lambda_3, s_3)$, then $(\lambda_1, s_1) \Rightarrow (\lambda_3, s_3)$. [6]

If measurements are unbiased, but not strongly unbiased, there still exists a covering of \Rightarrow s.t. within each element of the covering 1-5 hold. To establish $(\lambda_1, s_1) \Rightarrow (\lambda_2, s_2)$ iff there exists a path, \bar{s} , s.t. $\bar{s}(\lambda_1) = s_1$ and $\bar{s}(\lambda_2) = s_2$, it is sufficient to establish it within each element of the covering. As it turns out, statements 1-5 are sufficient for accomplishing this.

The key step is to show that if $(\lambda_0, s_0) \Rightarrow (\lambda_1, s_1)$ then there's a partial path, $\bar{s}[\lambda_0, \lambda_1]$ s.t. $\bar{s}[\lambda_0, \lambda_1](\lambda_0) = s_0$ and $\bar{s}[\lambda_0, \lambda_1](\lambda_1) = s_1$. If Λ is discrete this can be readily shown using statement 5 and induction. The proof when Λ a continuum will now be briefly sketched. First, for each $\lambda_0 < \lambda < \lambda_1$, form the set of all s s.t. $(\lambda_0, s_0) \Rightarrow (\lambda, s)$ and $(\lambda, s) \Rightarrow (\lambda_1, s_1)$. For $\lambda_{1/2} \equiv \frac{1}{2}(\lambda_0 + \lambda_1)$ choose a $s_{1/2}$ from $\lambda_{1/2}$'s set. Now for each $\lambda_0 < \lambda < \lambda_{1/2}$ form the set of all s s.t. $(\lambda_0, s_0) \Rightarrow (\lambda, s)$ and $(\lambda, s) \Rightarrow (\lambda_{1/2}, s_{1/2})$, and for each $\lambda_{1/2} < \lambda < \lambda_1$ form the set of all s s.t. $(\lambda_{1/2}, s_{1/2}) \Rightarrow (\lambda, s)$ and $(\lambda, s) \Rightarrow (\lambda_1, s_1)$. For $\lambda_{1/4} \equiv \frac{1}{2}(\lambda_0 + \lambda_{1/2})$ and $\lambda_{3/4} \equiv \frac{1}{2}(\lambda_{1/2} + \lambda_1)$, choose a $s_{1/4}$ and a $s_{3/4}$ from the newly formed sets and repeat the process. When this has been done for all $\lambda = m/2^n$, at each of the other λ 's take the intersection of all the formed sets, and select one element (the intersection must be non-empty). The set of all the selected $(\lambda_{m/2^n}, s_{m/2^n})$ and all the (λ, s) 's chosen from the intersections is the graph of a partial path running from (λ_0, s_0) to (λ_1, s_1) .

Just as as one can start with the transition relation and use it to define a set of paths, one can also start with a set of paths, and from it derive the transition relation: If S is a dynamic set, $(\lambda_1, s_1) \Rightarrow (\lambda_2, s_2)$ if $S_{(\lambda_1, s_1) \rightarrow (\lambda_2, s_2)} \neq \emptyset$. With the \Rightarrow relation defined in this way, 1-4 above will always hold; if S is a dynamic space, 5 will also hold. However, if the dynamic set is not a dynamic space, the two representations may not be equivalent, paths can contain more information. To see this, consider the case where $(\lambda_1, s_1) \Rightarrow (\lambda_2, s_2)$, $(\lambda_2, s_2) \Rightarrow (\lambda_3, s_3)$,

and $(\lambda_1, s_1) \Rightarrow (\lambda_3, s_3)$; this says that a path runs from (λ_1, s_1) to (λ_2, s_2) , a path runs from (λ_2, s_2) to (λ_3, s_3) , and a path run from (λ_1, s_1) to (λ_3, s_3) , but it doesn't guarantee that any individual path runs through all three points. However, when constructing paths from the transition relation, it might be possible to construct such a path. Therefore, starting with a dynamic set S , using S to construct the \Rightarrow relation, then using \Rightarrow to construct the set of paths, S' , you can have $S \subsetneq S'$, the elements of $S' - S$ being non-existent paths that can't be ruled out based on the transition relation alone.

From here on out we will revert to the path formalism.

2. Discretely Determined Partitions

The mathematical framework employed in quantum mechanics limits the types of measurements that the theory can talk about. These limitations are of essentially two types. First, because the probabilities are defined directly on the outcomes, rather than on the sets of outcomes, partitions have to be at most countable. Second, because measurements are described by projection operators on the state space, experimental outcomes correspond to sequences of measurements of $S(\lambda)$ at discrete values of Λ_S (S being the system's dynamic set). The following definitions will allow us to work within these constraints.

Definition 145. For dynamic set, S , and $L \subset \Lambda_S$:

$\alpha \subset S$ is *determined on L* if $\alpha = \bigcap_{\lambda \in L} S_{\rightarrow(\lambda, \alpha(\lambda)) \rightarrow} = \{\bar{p} \in S : \text{for all } \lambda \in L, \bar{p}(\lambda) \in \alpha(\lambda)\}$ (essentially, $\alpha = S_{\rightarrow(\lambda_1, \alpha(\lambda_1)) \rightarrow \dots \rightarrow (\lambda_i, \alpha(\lambda_i)) \rightarrow \dots}$)

If γ is a partition of S , γ is determined on L if all $\alpha \in \gamma$ are determined on L .

α is *discretely determined* if it is determined on some discrete L ; similarly for partitions.

Note that if α is determined on L , and $L \subset L'$, then α is determined on L' .

Quantum probabilities are calculated using equations of the form $P(A_1, A_2, \dots | S) = \langle S, \lambda_0 | \mathbb{P}(A_1; \lambda_1) \mathbb{P}(A_2; \lambda_2) \dots \mathbb{P}(A_n; \lambda_n) \mathbb{P}(A_n; \lambda_n) \dots \mathbb{P}(A_2; \lambda_2) \mathbb{P}(A_1; \lambda_1) | S, \lambda_0 \rangle$ where S is the initial system state and $\mathbb{P}(A_i; \lambda_i)$ is the projection operator onto state space region A_i at λ_i . This manner of calculation necessarily limits the formalism to partitions composed of discretely determined measurements. Before moving on, let's briefly take a closer look at the nature of this discreteness.

To be able to calculate (or even represent) $|\psi\rangle \equiv \dots \mathbb{P}(A_i; \lambda_i) \dots \mathbb{P}(A_2; \lambda_2) \mathbb{P}(A_1; \lambda_1) | S, \lambda_0 \rangle$,

$L \equiv \{\lambda_1, \lambda_2, \dots, \lambda_i, \dots\}$ must have a least element (which should not be less than λ_0), and for every $\lambda \in L$, the set of elements of L that are greater than λ must have a least element; in other words, L must be well ordered. Under these conditions, we have $|\psi_1 \rangle \equiv \mathbb{P}(A_1; \lambda_1)|S, \lambda_0 \rangle$, $|\psi_2 \rangle \equiv \mathbb{P}(A_2; \lambda_2)|\psi_1 \rangle$, etc. If the sequence is finite, terminating at n , then $|\psi \rangle = |\psi_n \rangle$. Otherwise, $|\psi \rangle$ is the limit of the sequence. The required probability is then $\langle \psi | \psi \rangle$.

Similarly, to be able to calculate $O \equiv \mathbb{P}(A_1; \lambda_1)\mathbb{P}(A_2; \lambda_2)\dots\mathbb{P}(A_i; \lambda_i)\dots\mathbb{P}(A_i; \lambda_i)\dots\mathbb{P}(A_2; \lambda_2)\mathbb{P}(A_1; \lambda_1)$, $L \equiv \{\lambda_1, \lambda_2, \dots, \lambda_i, \dots\}$ must have a greatest element, and for every $\lambda \in L$, the set of elements of L that are less than λ must have a greatest element; in other words, L must be “upwardly well ordered”. Under these conditions, the probability is $\langle S, \lambda_0 | O | S, \lambda_0 \rangle$.

Because we are interested in partitions with total probability that’s guaranteed to be 1 based only on the structure of the measurements, and independent of the details of the inner-products, we are limited to partitions that are determined on parameter sets of this second type. It should be noted, however, that this further qualification is of little consequence, because all finitely determined partitions are allowed, and countable cases can be taken as the limit of a sequence of finite cases.

Definition 146. If L is the subset of a parameter, it is *upwardly well-ordered* if every non-empty subset of L has a greatest element.

If γ is a partition of S , $\gamma \in \mathfrak{T}_S$ if γ is countable and for some upwardly well-ordered, bounded from below $L \subset \Lambda_S$, γ is determined on L .

If $\gamma \in \mathfrak{T}_S$, $L(\gamma)$ is the set of upwardly well-ordered, bounded from below sets, $l \subset \Lambda$, s.t. γ is determined on l .

From here on out, “discretely determined” will refer to being determined on an L that’s upwardly well-ordered and bounded from below.

Regardless of how “discrete” is interpreted, being limited to discretely determined measurements is a surprisingly strong constraint. To see this, consider the following simple type of measurement:

Definition 147. If S is a dynamic space and $\alpha \subset S$, α is a *moment* if for all $\lambda \in \Lambda_S$, α is either $bb\lambda$ or $ba\lambda$.

Assume that α is $bb\lambda$ and for all $\lambda' > \lambda$, α is $ba\lambda'$. This makes α a moment. If α is $ba\lambda$ then α is a measurement of the system state at λ . Otherwise, if α is not $ba\lambda$, α may be

thought of as a measurement of the system's state and its rate of change. Colloquially, it may be thought of as a measurement of position and velocity[7]. One limitation of the Hilbert Space formulation of quantum physics is that it can only be used to describe measurements of system state alone.

Theorem 148. *If X is a pairwise-disjoint collection of moments of a dynamic space, then X is compatible.*

Proof. Moments of a dynamic space are clearly companionable. Since X is pairwise-disjoint, it only remains to show that for all $\alpha \in X$, if $\beta \in (\alpha)_\lambda^X$ and $p \in \alpha(\lambda) \cap \beta(\lambda)$ then $\alpha_{\rightarrow(\lambda,p)} = \beta_{\rightarrow(\lambda,p)}$.

A: If $(\alpha)_\lambda^X \neq \{\alpha\}$ then α is $bb\lambda$.

- Assume $\alpha[-\infty, \lambda] \cap \beta[-\infty, \lambda] \neq \emptyset$ and $\alpha \neq \beta$. If α is $ba\lambda$ then $\alpha \cap \beta \neq \emptyset$ which contradicts X being pairwise-disjoint. Since α is a moment, and is not $ba\lambda$, it is $bb\lambda$. -

Assume $\beta \in (\alpha)_\lambda^X$ and $\beta \neq \alpha$, then by (A), α is $bb\lambda$. Similarly, because $\alpha \in (\beta)_\lambda^X$, β is $bb\lambda$. Therefore, for any $p \in \alpha(\lambda) \cap \beta(\lambda)$, $\alpha_{\rightarrow(\lambda,p)} = \beta_{\rightarrow(\lambda,p)}$. \square

It follows that any partition composed of moments is an ip. We can reasonably assume that all moment measurements can be performed; indeed, they appear to be quite useful. However, they can not all be represented using the quantum formalism. This represents a rather severe limitation in the mathematical language of quantum physics.

In spite of such limitations, there is one way in which \mathcal{T}_S may be considered overly inclusive. Dynamic sets are a very broad concept, which can make them unwieldy to use. To reign in their unruliness, we generally assume that they can be decomposed into ip's, and that these ip's are related to the possible measurement on the set. The subset of \mathcal{T}_S containing elements that support to such decompositions will prove useful.

Definition 149. If S is a dynamic set, Γ_S is the set of $\gamma \in \mathcal{T}_S$ s.t. for every $\alpha \in \gamma$ there's an ip of S , ς , s.t. for some $\nu \in \varsigma$, $\alpha \subset \nu$.

If D is a dynamic space then $\Gamma_S = \mathcal{T}_S$. In the quantum formalism, when $\Gamma_S \subsetneq \mathcal{T}_S$, it is Γ_S that is of interest, for if α is an outcome of a quantum measurement, then it must be an element of an ip. To see why, start with the a time ordered product of projection operators corresponding to the measurement, $...\mathbb{P}(A_i; \lambda_i)...\mathbb{P}(A_2; \lambda_2)\mathbb{P}(A_1; \lambda_1)$. Define $\mathbb{P}(\neg A_j; \lambda_j) \equiv$

$\mathbb{I} - \mathbb{P}(A_j; \lambda_j)$; with X_j equal to either A_j or $\neg A_j$, form the set of all time ordered products of the form $\dots \mathbb{P}(X_i; \lambda_i) \dots \mathbb{P}(X_2; \lambda_2) \mathbb{P}(X_1; \lambda_1)$. This set corresponds to an ip.

Theorem 150. *If S is a dynamic set and $\alpha \in \gamma \in \Gamma_S$ then α is companionable*

Proof. Take ς to be an ip and $\alpha \subset \nu \in \varsigma$. If $\bar{p}_1, \bar{p}_2 \in \alpha$ and $\bar{p}_1(\lambda) = \bar{p}_2(\lambda)$ then $\bar{p} = \bar{p}_1[-\infty, \lambda] \circ \bar{p}_2[\lambda, \infty] \in \nu$. Take any $L \in L(\alpha)$. For all $\lambda \in L$, $\bar{p}(\lambda) \in \alpha(\lambda)$, so $\bar{p} \in \alpha$, and so α is a subspace. With $\lambda_0 = glb(L)$, there must be a $\lambda \leq \lambda_0$ s.t. ν is $sbb\lambda$. Since α is determined on L , α is then also $sbb\lambda$. Taking λ_1 to be L 's greatest element, α is $wba\lambda_1$. \square

Theorem 151. *If S is a dynamic set then given any $\alpha, \beta \in \bigcup \Gamma_S$ s.t. $\alpha_{\rightarrow(\lambda,p)} \cap \beta_{\rightarrow(\lambda,p)} \neq \emptyset$, if $\bar{p}_1[-\infty, \lambda] \in \alpha_{\rightarrow(\lambda,p)}$ and $\bar{p}_2[\lambda, \infty] \in \beta_{(\lambda,p) \rightarrow}$ then $\bar{p}_1[-\infty, \lambda] \circ \bar{p}_2[\lambda, \infty] \in S$.*

Proof. Take $\bar{p}[-\infty, \lambda] \in \alpha_{\rightarrow(\lambda,p)} \cap \beta_{\rightarrow(\lambda,p)}$, $\bar{p} \equiv \bar{p}[-\infty, \lambda] \circ \bar{p}_2[\lambda, \infty]$. Since β is a subspace, $\bar{p} \in S$. Take ς to be an ip and $\alpha \subset \nu \in \varsigma$. For some $\eta \in \gamma$, $\bar{p} \in \eta$. $\bar{p}[-\infty, \lambda] \in \nu_{\rightarrow(\lambda,p)} \cap \eta_{\rightarrow(\lambda,p)}$, so $\eta_{\rightarrow(\lambda,p)} = \nu_{\rightarrow(\lambda,p)}$, and so $\bar{p}_1[-\infty, \lambda] \in \eta_{\rightarrow(\lambda,p)}$. Since η is a subspace and $\bar{p}[\lambda, \infty] = \bar{p}_2[\lambda, \infty] \in \eta_{(\lambda,p) \rightarrow}$, $\bar{p}_1[-\infty, \lambda] \circ \bar{p}_2[\lambda, \infty] \in \eta \subset S$. \square

3. Interconnected Dynamic Sets

An all-reet e-automata can not forget anything its ever known about the system. Under the right conditions, however, an e-automata can discover something about the system's past that is not implied by the current state. This is most readily seen if the system parameter is discrete.

For an e-automata with a discrete parameter, assume that at λ the e-automata is in state (s_λ, e_λ) and at $\lambda + 1$ it's in state $(s_{\lambda+1}, e_{\lambda+1})$. It would be reasonable (though not necessary) to assume that the system state of s_λ is not reflected in e_λ , and doesn't get reflected in the environment until $e_{\lambda+1}$. In this case, it takes one step in time for the environment to learn about the system. More generally, of course, the set of allowed environmental states at $\lambda + 1$ will be a function of both s_λ and $s_{\lambda+1}$, as well as e_λ , but this still asserts that it is possible to learn something about the system's past that's not reflected in the current state of the system.

If the parameter is continuous and the measurements are at discrete times we wouldn't expect this to happen, though it can; the environment may gain information at λ_2 about

the state of the system at λ_1 that isn't implied by the state of the system at λ_2 . In order for this to occur the system itself would have to remember something about its state at λ_1 until λ_2 , at which point the knowledge is simultaneously passed to the environment and forgotten by the system. This is clearly an edge case, but a pernicious one, as it allows for anomalous ip's with discretely determined measurements; in order to regularize this set of ip's, this edge case needs to be eliminated. In order to be eliminated, the system dynamics need to be interconnected, a property that will be defined first for dynamic spaces, and then for dynamic sets.

Definition 152. A dynamic space, D , is *interconnected* if

for all $\lambda_1 < \lambda_2$, every $p_1, p'_1 \in D(\lambda_1)$, $p_2, p'_2 \in D(\lambda_2)$ s.t. $(\lambda_1, p_1) \rightarrow (\lambda_2, p_2) \neq \emptyset$, $(\lambda_1, p_1) \rightarrow (\lambda_2, p'_2) \neq \emptyset$, $(\lambda_1, p'_1) \rightarrow (\lambda_2, p_2) \neq \emptyset$, and $(\lambda_1, p'_1) \rightarrow (\lambda_2, p'_2) \neq \emptyset$,
there exists a $\lambda \in [\lambda_1, \lambda_2]$ and a $p \in D(\lambda)$ s.t. $(\lambda_1, p_1) \rightarrow (\lambda, p) \neq \emptyset$, $(\lambda_1, p'_1) \rightarrow (\lambda, p) \neq \emptyset$, $(\lambda, p) \rightarrow (\lambda_2, p_2) \neq \emptyset$, and $(\lambda, p) \rightarrow (\lambda_2, p'_2) \neq \emptyset$.

Being interconnected is equivalent to saying that if $p_1 \neq p'_1$ then $\rightarrow \{(\lambda_1, p_1), (\lambda_1, p'_1)\} \rightarrow (\lambda_2, p_2) \rightarrow, \rightarrow (\lambda_1, p_1) \rightarrow (\lambda_2, p'_2) \rightarrow$, and $\rightarrow (\lambda_1, p'_1) \rightarrow (\lambda_2, p'_2) \rightarrow$ can not be mutually compatible. Essentially, for any $\bar{p}[-\infty, \lambda_1] \in \rightarrow (\lambda_1, p_1)$, $\bar{p}'[-\infty, \lambda_1] \in \rightarrow (\lambda_1, p'_1)$, if $\bar{p}[-\infty, \lambda_1]$ and $\bar{p}'[-\infty, \lambda_1]$ have not been distinguished by λ_1 , and there is no measurement between λ_1 and λ_2 , then they can not be distinguished at λ_2 because the point (λ, p) destroys the ability to distinguish them. Nearly all actively studied dynamic systems are interconnected, and quantum systems always are.

Now to generalize interconnectedness for dynamic sets.

Definition 153. If S is a dynamic set and $\bar{p}_{11}, \bar{p}_{21}, \bar{p}_{12}, \bar{p}_{22} \in S$, $(\bar{p}_{11}, \bar{p}_{21}, \bar{p}_{12}, \bar{p}_{22})$ is an *interconnect on* $[\lambda_1, \lambda_2]$ if

$$\begin{aligned}\bar{p}_{11}[-\infty, \lambda_1] &= \bar{p}_{12}[-\infty, \lambda_1] \\ \bar{p}_{21}[-\infty, \lambda_1] &= \bar{p}_{22}[-\infty, \lambda_1] \\ \bar{p}_{11}[\lambda_2, \infty] &= \bar{p}_{21}[\lambda_2, \infty] \\ \bar{p}_{12}[\lambda_2, \infty] &= \bar{p}_{22}[\lambda_2, \infty].\end{aligned}$$

The paths $\bar{p}_{11}, \bar{p}_{21}, \bar{p}_{12}, \bar{p}_{22}$ can be thought of as sample paths from the sets $\rightarrow (\lambda_1, p_1) \rightarrow (\lambda_2, p_2) \rightarrow$, etc., that were used in the dynamic space definition of interconnectedness. Because e-automata are unbiased, if there are no measurements on these paths between λ_1 and λ_2 then $\rightarrow \{\bar{p}_{11}[-\infty, \lambda_1], \bar{p}_{21}[-\infty, \lambda_1]\}$ should behave like a dynamic space in $[\lambda_1, \lambda_2]$.

Definition 154. If A is a dynamic set, $[\lambda_1, \lambda_2]$ is a *space-segment* of A if for all $\bar{p} \in A$, $[\lambda, \lambda'] \subset [\lambda_1, \lambda_2]$, $\bar{p}'[\lambda, \lambda'] \in A_{(\lambda, \bar{p}(\lambda)) \rightarrow (\lambda', \bar{p}(\lambda'))}$, $\bar{p}[-\infty, \lambda] \circ \bar{p}'[\lambda, \lambda'] \circ \bar{p}[\lambda', \infty] = A$.

If an experiment has not distinguished between the paths of an interconnect on $[\lambda_1, \lambda_2]$ by λ_1 , and there is no measurement between λ_1 and λ_2 , then interconnectedness will insure that they can not be distinguished at λ_2 :

Definition 155. A dynamic set, S , is *interconnected* if for every $\lambda_1 < \lambda_2$, every interconnect on $[\lambda_1, \lambda_2]$, $(\bar{p}_{11}, \bar{p}_{21}, \bar{p}_{12}, \bar{p}_{22})$, s.t. $[\lambda_1, \lambda_2]$ is a space-segment of $+\{\bar{p}_{11}[-\infty, \lambda_1], \bar{p}_{21}[-\infty, \lambda_1]\}$ there exists a $\lambda \in [\lambda_1, \lambda_2]$ and an interconnect on $[\lambda, \lambda_2]$, $(\bar{p}'_{11}, \bar{p}'_{21}, \bar{p}'_{12}, \bar{p}'_{22})$, s.t. $\bar{p}'_{ij}[-\infty, \lambda_1] = \bar{p}_{ij}[-\infty, \lambda_1]$, $\bar{p}'_{ij}[\lambda_2, \infty] = \bar{p}_{ij}[\lambda_2, \infty]$, and all $\bar{p}'_{ij}(\lambda) = \bar{p}_{mn}(\lambda)$.

$[\lambda_1, \lambda_2]$ being a space-segment of $+\{\bar{p}_{11}[-\infty, \lambda_1], \bar{p}_{21}[-\infty, \lambda_1]\}$ helps to insure that there exists an interconnect, $(\bar{p}'_{11}, \bar{p}'_{21}, \bar{p}'_{12}, \bar{p}'_{22})$, rather than just four paths that share the same point. If S is a dynamic space, this definition is equivalent to the prior one.

Finally, to make interconnectedness more readily applicable to Γ_S for the case when the system is not a dynamic space.

Definition 156. If $L \subset \Lambda$ is upwardly well-ordered, and λ is not a lower-bound of L , $pred_L(\lambda) \equiv lub(L \cap [-\infty, \lambda))$.

$pred_L(\lambda)$ is short for “predecessor of λ on L ”.

Definition 157. For $\gamma \in \Gamma_S$, $\mathcal{L}(\gamma) \subset L(\gamma)$ s.t. if $L \in \mathcal{L}(\gamma)$ then for all $\alpha \in \gamma$, $\lambda \in L - \{glb(L)\}$, $[pred_L(\lambda), \lambda]$ is a space-segment of $+\alpha[-\infty, pred_L(\lambda)]$.

$\gamma \in \Gamma_S^s$ if $\gamma \in \Gamma_S$ and $\mathcal{L}(\gamma) \neq \emptyset$.

For quantum systems, the dynamics between successive measurements is always a space-segment, and so when discussing measurements only the elements of Γ_S^s are of interest. For dynamic spaces, $\Gamma_S^s = \Gamma_S = \mathbf{\Upsilon}_S$.

B. Partitions of Unity for Quantum Systems

A partition with total probability of 1 will be given the fancy title of being a *partition of unity*. In this section, we consider the quantum partitions of unity.

When discussing quantum probabilities there are two cases to be considered: the conditional case, where an initial state is known, and the non-conditional case, where no initial

state is known & everything about the system is discovered via the experiment. We will start by considering the conditional case.

1. The Conditional Case

Definition 158. If γ is a partition of dynamic set S and $\bar{p}_1, \bar{p}_2 \in S$, \bar{p}_1 and \bar{p}_2 are *co-located* on γ if there's a $\alpha \in \gamma$ s.t. $\bar{p}_1, \bar{p}_2 \in \alpha$; \bar{p}_1 and \bar{p}_2 *converge at λ* if $\bar{p}_1[\lambda, \infty] = \bar{p}_2[\lambda, \infty]$

The central object of study in this section will be the set of partitions q_S :

Definition 159. For dynamic set S , $\gamma \in q_S$ if $\gamma \in \Gamma_S^s$ and for some $L \in \mathcal{L}(\gamma)$, all $\lambda_1, \lambda_2 \in L$ s.t. $\lambda_1 \equiv \text{pred}_L(\lambda_2)$, all $\bar{p}_1[-\infty, \lambda_1], \bar{p}_2[-\infty, \lambda_1] \in S[-\infty, \lambda_1]$, either all $\bar{p} \in +\bar{p}_1[-\infty, \lambda_1]$, $\bar{p}' \in +\bar{p}_2[-\infty, \lambda_1]$ that converge at λ_2 are co-located on γ , or none are.

In the following claim, total probabilities of a set of outcomes are “guaranteed to equal 1” if it's known to be 1 based only on the structure of the outcomes, and independent of the details of the transition probabilities, including the choice of initial state.

Claim 160. For quantum probabilities in the conditional case, the sum over probabilities of a set of outcomes is guaranteed to equal 1 iff the set of outcomes is an element of q_S .

Remark. It has already been argued that quantum outcomes are elements of $\bigcup \Gamma_S^s$. The rest of the claim may be justified as follows.

Quantum outcomes are represented by time ordered products of projection operators. Let's take L to be the set of times at which a projection operator is applied, and define λ_1 to be the largest element of L , λ_2 to be the largest element of $L - \{\lambda_0\}$, etc. (Note that as the subscript increases, the time decreases.) An outcome is then represented by $\Pi_i = \dots \mathbb{P}(A_{ij}; \lambda_j) \dots \mathbb{P}(A_{i1}; \lambda_1)$, where $A_{ij} \subset S(\lambda_j)$, the “ i ” subscript identifies the outcome, and the “ j ” subscript identifies the time. If the set of outcomes form a partition then $\sum_i \Pi_i = \mathbb{I}$ (\mathbb{I} being the identity operator). This yields a total probability of 1 irregardless of the choice of the initial state iff $\sum_i \langle \psi, \lambda_0 | \Pi_i \Pi_i^\dagger | \psi, \lambda_0 \rangle = 1$ for all initial states $|\psi, \lambda_0 \rangle$, which holds iff $\sum_i \Pi_i \Pi_i^\dagger = \mathbb{I}$.

For any $\lambda_j \in L$, define χ_j by $(\dots, s_k, \dots, s_j) \in \chi_j$ if for all $\lambda_q < \lambda_r \leq \lambda_j$ ($\lambda_q, \lambda_r \in L$) the transition $(s_q, \lambda_q) \Rightarrow (s_r, \lambda_r)$ is allowed;

Further, define Ξ_j by $(\dots, s_k, s'_k, \dots, s_{j+1}, s'_{j+1}, s_j) \in \Xi_j$ if

- 1) $(..., s_k, ..., s_{j+1}, s_j) \in \chi_j$ and $(..., s'_k, ..., s'_{j+1}, s_j) \in \chi_j$
 - 2) For some outcome, $\Pi_i, s_j \in A_{ij}$ and for all $k > j$ (s.t. $\lambda_k \in L$) $s_k, s'_k \in A_{ik}$.
- $\sum_i \Pi_i \Pi_i^\dagger = \mathbb{I}$ can then be expanded to

Condition 1:

$$\int_{\Xi_1} ds_1 ds_2 ds'_2 ... |s_2, \lambda_2 \rangle \langle s_2, \lambda_2 | s_1, \lambda_1 \rangle \langle s_1, \lambda_1 | s'_1, \lambda_1 \rangle \langle s'_1, \lambda_1 | ... = \mathbb{I}$$

Start by integrating over s_1 ; the above identity can not hold unless for every $(..., s_2, s'_2) = (\tilde{s}, \tilde{s}') \in \text{Dom}(\Xi_1)$ either $\langle s_2, \lambda_2 | (\int_{(\tilde{s}, \tilde{s}', s_n) \in \Xi_1} ds_1 |s_1, \lambda_1 \rangle \langle s_1, \lambda_1 |) | s'_2, \lambda_2 \rangle = 1$ or $\langle s_2, \lambda_2 | (\int_{(\tilde{s}, \tilde{s}', s_n) \in \Xi_1} ds_1 |s_1, \lambda_1 \rangle \langle s_1, \lambda_1 |) | s'_1, \lambda_2 \rangle = 0$. When $\tilde{s} = \tilde{s}'$ the first equality clearly holds. For $\tilde{s} \neq \tilde{s}'$, the only way that we could have an s_1 s.t. $(\tilde{s}, s_1) \in \chi_1$, and $(\tilde{s}', s_1) \in \chi_1$, but $(\tilde{s}, \tilde{s}', s_1) \notin \Xi_1$, and still have one of these equalities hold is if either $\langle s_2, \lambda_2 | s_1, \lambda_1 \rangle = 0$ or $\langle s'_2, \lambda_2 | s_1, \lambda_1 \rangle = 0$ (this makes the claim of $(\tilde{s}, s_1) \in \chi_1$, and $(\tilde{s}', s_1) \in \chi_1$ uncomfortable, but the situation can not be ruled out). Since we are interested in the conditions under which the probabilities are guaranteed to sum to 1 regardless of the details of inner-products, we eliminate this last case and are lead to:

Condition 2: If $(\tilde{s}, \tilde{s}') \in \text{Dom}(\Xi_1)$ then for all s_1 s.t. $(\tilde{s}, s_1) \in \chi_1$ and $(\tilde{s}', s_1) \in \chi_1$, $(\tilde{s}, \tilde{s}', s_1) \in \Xi_1$.

Which is equivalent to applying the q_S condition at λ_1 .

Apply Condition 2 to Condition 1, Condition 1 is reduced to:

Condition 3:

$$\int_{\Xi_1} ds_2 ds_3 ds'_3 ... |s_3, \lambda_3 \rangle \langle s_3, \lambda_3 | s_2, \lambda_2 \rangle \langle s_2, \lambda_2 | s'_3, \lambda_3 \rangle \langle s'_3, \lambda_3 | ... = \mathbb{I}$$

Which is identical in form to the Condition 1. Therefore, to satisfy Condition 1, it is sufficient to satisfy Condition 1, for all elements of L to satisfy Condition 2 (with the “1” subscript replaced with “ i ”). Note that Condition 2 was imposed as a necessary condition for Condition 1 to hold; we now see that it’s a necessary and sufficient condition. Finally, note that requiring Condition 2 on all λ_i is equivalent to requiring that the set of outcomes is an element of q_S .

Theorem 161. *For dynamic set S*

- 1) *If $\gamma \in q_S$ then γ is nearly compatible*
- 2) *If S is interconnected, $\gamma \in \Gamma_S^s$, and γ is nearly compatible then $\gamma \in q_S$*

Proof. By Thm 150, for all $\gamma \in \Gamma_S^s$, all $\alpha \in \gamma$ are companionable

1) For $\alpha, \beta \in \gamma$ assume $\alpha_{\rightarrow(\lambda,p)} \cap \beta_{\rightarrow(\lambda,p)} \neq \emptyset$, and $\alpha \neq \beta$. λ can not be an upper-bound on any $L(\gamma)$. Take any $\bar{p}[-\infty, \lambda] \in \alpha_{\rightarrow(\lambda,p)} \cap \beta_{\rightarrow(\lambda,p)}$, $\bar{p}'[-\infty, \lambda] \in \alpha_{\rightarrow(\lambda,p)}$, $\bar{p}_1[\lambda, \infty] \in \alpha_{(\lambda,p) \rightarrow}$, $\bar{p}_2[\lambda, \infty] \in \beta_{(\lambda,p) \rightarrow}$.

$$\bar{p}_1 \equiv \bar{p}[-\infty, \lambda] \circ \bar{p}_1[\lambda, \infty] \in \alpha$$

$$\bar{p}'_1 \equiv \bar{p}'[-\infty, \lambda] \circ \bar{p}_1[\lambda, \infty] \in \alpha$$

$$\bar{p}_2 \equiv \bar{p}[-\infty, \lambda] \circ \bar{p}_2[\lambda, \infty] \in \beta$$

$$\bar{p}'_2 \equiv \bar{p}'[-\infty, \lambda] \circ \bar{p}_2[\lambda, \infty] \text{ (Exists by Thm 151)}$$

Applying the definition of q_S to any $\lambda' \in L$ s.t. $\lambda' > \lambda$, it follows that $\bar{p}'_2 \in \beta$. Therefore $\alpha_{\rightarrow(\lambda,p)} = \beta_{\rightarrow(\lambda,p)}$.

2) Assume $\gamma \in \Gamma_S^s$ is nearly compatible. Take any $L \in \mathcal{L}(\gamma)$, any $\lambda_1 \in L$ s.t. $\lambda_1 \neq glb(L)$, and define $\lambda_0 \equiv pred_L(\lambda)$. Take any $\alpha \in \gamma$ and any $\bar{p}_1, \bar{p}_2 \in \alpha$ that converge at λ_1 (if they exist). Take $\bar{p}_3, \bar{p}_4 \in S$ such that $\bar{p}_3[-\infty, \lambda_0] = \bar{p}_1[-\infty, \lambda_0]$, $\bar{p}_4[-\infty, \lambda_0] = \bar{p}_2[-\infty, \lambda_0]$, and \bar{p}_3 and \bar{p}_4 converge at λ_1 . Further take $\bar{p}_3 \in \beta \in \gamma$. The theorem is proved if $\bar{p}_4 \in \beta$.

Because $(\bar{p}_1, \bar{p}_2, \bar{p}_3, \bar{p}_4)$ is an interconnect on $[\lambda_0, \lambda_1]$, and S is interconnected, there's a $\lambda' \in [\lambda_0, \lambda_1]$ and $\bar{p}'_1, \bar{p}'_2, \bar{p}'_3, \bar{p}'_4 \in S$ s.t. $\bar{p}'_i[-\infty, \lambda_0] = \bar{p}_i[-\infty, \lambda_0]$, $\bar{p}'_i[\lambda_1, \infty] = \bar{p}_i[\lambda_1, \infty]$, $\bar{p}'_1[-\infty, \lambda'] = \bar{p}'_3[-\infty, \lambda']$, $\bar{p}'_2[-\infty, \lambda'] = \bar{p}'_4[-\infty, \lambda']$, and all $\bar{p}'_i(\lambda') = \bar{p}'_j(\lambda') = p$.

Since \bar{p}'_i and \bar{p}_i are equal on L , and all elements of γ are determined on L , they are co-located on γ . Therefore $\bar{p}'_1, \bar{p}'_2 \in \alpha$ and $\bar{p}'_3 \in \beta$. Since $\bar{p}'_1[-\infty, \lambda'] = \bar{p}'_3[-\infty, \lambda'] \in \alpha_{\rightarrow(\lambda',p)} \cap \beta_{\rightarrow(\lambda',p)}$ and γ is nearly compatible, $\alpha_{\rightarrow(\lambda',p)} = \beta_{\rightarrow(\lambda',p)}$. Since $\bar{p}'_4[-\infty, \lambda'] = \bar{p}'_2[-\infty, \lambda'] \in \alpha_{\rightarrow(\lambda',p)}$, $\bar{p}'_4[-\infty, \lambda'] \in \beta_{\rightarrow(\lambda',p)}$. Since $\bar{p}'_4[\lambda_1, \infty] = \bar{p}_4[\lambda_1, \infty] = \bar{p}_3[\lambda_1, \infty]$, for all $\lambda \in L$, $\bar{p}'_4(\lambda) \in \beta(\lambda)$, so $\bar{p}'_4 \in \beta$. Since \bar{p}_4 and \bar{p}'_4 are equal on all $\lambda \in L$, $\bar{p}_4 \in \beta$. \square

2. The Non-Conditional Case

For the non-conditional case, no initial state is assumed. Once again representing each outcome, Π_i , as a product of projection operators, $\Pi_i = \dots \mathbb{P}(A_{ij}; \lambda_j) \dots \mathbb{P}(A_{i1}; \lambda_1)$, the probability of a given outcome is $P_i = \frac{1}{Tr(\mathbb{I})} Tr(\Pi_i \Pi_i^\dagger)$ (ignoring complications that arise if $Tr(\mathbb{I})$ is infinite). This can be arrived at by assuming that, if the initial state is unknown, then the probability is the average for all possible initial states: Given any initial state, (s, λ_0) , the probability for obtaining outcome i is $P_{i|(s, \lambda_0)} = \langle s, \lambda_0 | \Pi_i \Pi_i^\dagger | s, \lambda_0 \rangle$. With V the volume of $S(\lambda_0)$, the average probability is then $\frac{1}{V} \sum_{s \in S(\lambda_0)} \langle s, \lambda_0 | \Pi_i \Pi_i^\dagger | s, \lambda_0 \rangle$. Since $V = Tr(\mathbb{I})$

and $\sum_{s \in S(\lambda_0)} < s, \lambda_0 | \Pi_i \Pi_i^\dagger | s, \lambda_0 > = \text{Tr}(\Pi_i \Pi_i^\dagger)$, this average probability is equal to P_i given above.

In the non-conditional case, the total probability is therefore guaranteed to be 1 iff $\sum_i \text{Tr}(\Pi_i \Pi_i^\dagger) = \text{Tr}(\mathbb{I})$.

Note that the assumption that the non-conditional probability is equal to the average conditional probabilities need not hold for a dps; there is nothing in the nature of the dps that demands it. The assumption is equivalent to a certain kind of additivity: Take α to be an outcome that's $bb\lambda$, define $x \equiv \alpha(\lambda)$, and select x_1, x_2 s.t. $x_1 \cup x_2 = x$ and $x_1 \cap x_2 = \emptyset$; define $\alpha_1 \equiv \alpha \cap S_{\rightarrow(\lambda, x_1)}$ and $\alpha_2 \equiv \alpha \cap S_{\rightarrow(\lambda, x_2)}$; the assumption being made is that under such circumstances, $P(\alpha) = P(\alpha_1) + P(\alpha_2)$.

We would expect this to hold if α is bounded from above by λ , but not if it's bounded from below. To see that it will hold when bounded from above, assume α is $ba\lambda$ (so the measurement of $\{x_1, x_2\}$ would be the final measurement, rather than the initial one), take γ to be any ip s.t. $\alpha \in \gamma$, and define $A \equiv \gamma - \{\alpha\}$. $A \cup \{\alpha_1, \alpha_2\}$ is an ip. If $P(A)$ is unchanged by whether A is paired with α or $\{\alpha_1, \alpha_2\}$ then $P(\alpha) = P(\alpha_1) + P(\alpha_2)$. Call this property “additivity of final state”; call the bounded from below case “additivity of initial state”.

Achieving additivity of initial state is not quite as straightforward as additivity final of state. One obvious way to do so is to assume additivity of initial state and reversibility. There are, however, other ways for the behavior to be realized. (See Appendix C for the definition of reversibility on a DPS.)

C. Maximal Quantum Systems

The foundations of quantum physics may be thought of as being composed of three interconnected parts: measurement theory, probability theory, and probability dynamics. This article has largely been concerned with creating a mathematical language sufficient for utilization in the measurement theory & that portion of the probability theory implied by the measurement theory. One does not expect a scientific theory to follow immediately from the mathematical language that it uses, unless the theory is fairly trivial. For this reason, Thm 161 is quite evocative.

Thm 161 says an interconnected dynamic system will possess the basic character of a quantum system if the finite, discretely-determined portion of its dps can be embedded in a

Υ -maximal dps. The question now is, under what conditions is such an embedding possible?

All countable, discretely-determined ip's will have consistent probabilities if the system is countably additive in its final state. As mentioned earlier, it may be assumed that a dynamic system has this property. To see why it is sufficient, start by choosing any λ , and any countable partition of $S(\lambda)$. By assumption, the total probability of the associated outcomes will be 1. For any of these outcome, optionally choose any $\lambda' > \lambda$, and any countable partition of $S(\lambda')$; by assumption, the total probability of these new outcomes will equal the probability of the original outcome. Continuing in this way, one can create all countable, discretely-determined ip's of the dynamic space.

To understand when such a dps could be maximal, consider a simple case of a set of outcomes that are nearly compatible, but not compatible. Assume there are $p_{1,0}, p_{1,1}, p_{1,2} \in S(\lambda_1)$ and $p_{2,0}, p_{2,1}, p_{2,2} \in S(\lambda_2)$ ($\lambda_1 < \lambda_2$) s.t. each of the $p_{2,j}$'s can be reached from $(\lambda_1, p_{1,j})$ and $(\lambda_1, p_{1,(j+1) \bmod 3})$, but not $(\lambda_1, p_{1,(j+2) \bmod 3})$. For $p_{2,0}$ and $p_{2,1}$ form outcomes consisting of both of the $p_{1,i}$ that can reach them: $O_0 = (\lambda_1, \{p_{1,0}, p_{1,1}\}) \rightarrow (\lambda_2, p_{2,0})$ and $O_1 = (\lambda_1, \{p_{1,1}, p_{1,2}\}) \rightarrow (\lambda_2, p_{2,1})$. For $p_{2,2}$ form outcomes from $p_{1,2}$ and $p_{1,0}$ individually: $O_{21} = (\lambda_1, p_{1,2}) \rightarrow (\lambda_2, p_{2,2})$ and $O_{22} = (\lambda_1, p_{1,0}) \rightarrow (\lambda_2, p_{2,2})$. The collection of these 4 outcomes is nearly compatible, but not compatible. That they are nearly compatible can be seen from the fact that they satisfy the q_S condition; that they are not compatible can be seen from the fact that while the combination of the first two outcomes imply that only $\{p_{1,0}, p_{1,1}, p_{1,2}\}$ was measured at λ_1 , the second two imply that $p_{1,2}$ was distinguished from $p_{1,0}$ at λ_1 .

The inclusion of such sets of outcomes into a t-algebra would make probabilities more highly additive; in particular, it would weaken the non-additivity of double-slit experiments. To see this, take the simplest case of $S(\lambda_i) = \{p_{i,0}, p_{i,1}, p_{i,2}\}$; because $\{O_0, O_1, O_{21} \cup O_{22}\}$ is compatible, and would certainly be on the t-algebra, we'd then have $P(O_{21} \cup O_{22}) = P(O_{21}) + P(O_{22})$, rendering the probabilities of this particular double-slit experiment entirely additive. However, if there exists a $p \in S(\lambda_2)$ that can be reached by $p_{1,0}$, $p_{1,1}$, and $p_{1,2}$, then the q_S condition entails that there can be no element of q_S that contains O_0 , O_1 , O_{21} , and O_{22} ; in this case a maximal t-algebra can not include $\{O_0, O_1, O_{21}, O_{22}\}$, and so will not demand $P(O_{21} \cup O_{22}) = P(O_{21}) + P(O_{22})$.

This illustrates how the finite & finitely determined portion of a dps can be maximal even if its probabilities are highly non-additive. If the paths of a dynamic system form a richly

interlocking network, then the elements of q_S that are not ip's will be limited, making it more likely that the system will be maximal. A rich network of paths will not limit the kinds of countable, discretely determined experiments that can be performed, it simply limits the amount of probabilistic information that can be extracted from them. A similar effect was seen previously when considering interconnectedness.

For non-deterministic systems, it is difficult to distinguish sets of paths that can not occur from sets of paths that occur with probability 0. If the approach to system dynamics is to include paths unless they can be ruled out in principal and let the probability function handle the rest, then the network of interlocking paths will be enriched. This will generally cause the discretely determined portion of such systems to be maximal. Let's see this in detail.

Definition 162. If γ_1 and γ_2 are partitions of some set, S , $\gamma_1 \prec \gamma_2$ if for all $\alpha \in \gamma_1$ there's a $\beta \in \gamma_2$ s.t. $\alpha \subset \beta$

As mentioned above, partitions with total probability of 1 can be iteratively created by taking any $\gamma \in \Gamma_S^s$ with probability known to be 1, and for each $\alpha \in \gamma$, select some λ s.t. α is $ba\lambda$, and slice up α by partitioning $\alpha(\lambda)$. However, not all partitions can be formed in this manner. If the partition is not compatible, and so can not be decided by an ideal e-automata, it is allowed to forget at λ some of what happened prior to λ . In such cases, to form the new partition, you wouldn't simply take the elements of γ and append measurements at λ , a further step would also be possible: for each measurement outcome at λ , multiple elements of γ may be combined into a single outcome. More precisely, select some countable set of $\gamma_i \succ \gamma$ such that each $\gamma_i \in \Gamma_S^s$. For every $p \in S(\lambda_n)$, select a γ_i . For every γ_i , every $\alpha \in \gamma_i$, create a countable partition of the set of $p \in S(\lambda_{n+1})$ that "selected" γ_i , $X_{i,\alpha}$, and for every $A \in X_{i,\alpha}$ form the new outcome $\alpha \rightarrow_{(\lambda_n, A)} \cdot$. The set of all such outcomes forms a new element of Γ_S^s . Under these general conditions, we can not comfortably assume that our new partition has a total probability of 1.

The q_S condition may be seen as a constraint on this formation process; whenever a $\gamma_i \succ \gamma$ is selected for one $p \in S(\lambda)$, the q_S condition constrains what may happen at the other elements of $S(\lambda)$. If this constraint implies that for all $\gamma \in q_S$, all λ s.t. γ is $ba\lambda$, all $p \in S(\lambda)$ must select the same $\gamma_i \succ \gamma$, and the selected γ_i must itself be an element of q_S , then we can expect all elements of q_S to have a total probability of 1.

To understand the effect that combining outcomes at one $p \in S(\lambda)$ has at the other points in $S(\lambda)$, it is sufficient to consider the combination of two outcomes. Since the two outcomes, $\alpha, \beta \in \gamma$, must form a discretely-determined set when combined into $\alpha \cup \beta$, they must be chosen so that for all but one element of L , $\alpha = \alpha_{\rightarrow(\lambda, \alpha(\lambda) \cup \beta(\lambda)) \rightarrow}$ and $\beta = \beta_{\rightarrow(\lambda, \alpha(\lambda) \cup \beta(\lambda)) \rightarrow}$; if $\alpha \neq \beta$, then $\alpha(\lambda)$ and $\beta(\lambda)$ must be disjoint at the one remaining $\lambda \in L$.

Now to see the effect of the q_S condition. Let's start by considering a pair of outcomes determined at a single λ , $S_{\rightarrow(\lambda_1, A) \rightarrow}$ and $S_{\rightarrow(\lambda_1, B) \rightarrow}$, and assume that at $p \in S(\lambda_2)$ these two are combined to form $S_{\rightarrow(\lambda_1, A \cup B) \rightarrow(\lambda_2, p) \rightarrow}$. The q_S condition demands that A and B must then also be combined for the outcomes at other elements of $S(\lambda)$. To see which ones, some further definitions will be helpful. (Note that $S_{(\lambda_1, A) \rightarrow(\lambda_2, p)}(\lambda_1)$ is the set of elements of A that can reach p .)

Definition 163. If S is a dynamic set, $\lambda_1 \leq \lambda_2$, $A \subset S(\lambda_1)$, $p \in S(\lambda_2)$ then

$$(A, \lambda_1) \downarrow (p, \lambda_2) \equiv S_{(\lambda_1, A) \rightarrow(\lambda_2, p)}(\lambda_1)$$

$$\text{For } X \subset S(\lambda_2), (A, \lambda_1) \downarrow (X, \lambda_2) \equiv S_{(\lambda_1, A) \rightarrow(\lambda_2, X)}(\lambda_1).$$

$(A, \lambda_1) \downarrow (p, \lambda_2)$ may be thought of as p 's footprint in A .

Definition 164. $p' \in [p, \lambda_2; A, \lambda_1]$ if $(A, \lambda_1) \downarrow (p', \lambda_2) \cap (A, \lambda_1) \downarrow (p, \lambda_2) \neq \emptyset$.

$p' \in [p, \lambda_2; A, \lambda_1]$ if p and p' 's footprints overlap, so $p' \in [p, \lambda_2; A, \lambda_1]$ if there exists an element of A that can reach both p and p' . According to the q_S condition, if outcomes A and B combine at p , and $p' \in [p, \lambda_2; A, \lambda_1] \cap [p, \lambda_2; B, \lambda_1]$, then they must combine at p' . If $p'' \in [p', \lambda_2; A, \lambda_1] \cap [p', \lambda_2; B, \lambda_1]$, they then must also combine at p'' . This leads to:

Definition 165. If S is a dynamic set, $\lambda_1 \leq \lambda_2$, $p \in S(\lambda_2)$, and A and B are disjoint subsets of $S(\lambda_1)$:

$$\|p, \lambda_2; A, B, \lambda_1\|_0 \equiv [\lambda_2, p; \lambda_1, A] \cap [\lambda_2, p; \lambda_1, B]$$

$$\|p, \lambda_2; A, B, \lambda_1\|_{n+1} \equiv \bigcup_{p' \in \|p, \lambda_2; A, B, \lambda_1\|_n} \|p', \lambda_2; A, B, \lambda_1\|_0$$

$$\|p, \lambda_2; A, B, \lambda_1\| \equiv \bigcup_{n \in \mathbb{N}} \|p, \lambda_2; A, B, \lambda_1\|_n$$

If A and B combine at p , then the q_S condition demands that they must combine at all elements of $\|p, \lambda_2; A, B, \lambda_1\|$.

There may be cases of $p' \in S(\lambda_2)$ s.t. for some $p_1 \in \|p, \lambda_2; A, B, \lambda_1\|$, $(A, \lambda_1) \downarrow (p', \lambda_2) \cap (A, \lambda_1) \downarrow (p_1, \lambda_2) \neq \emptyset$, and for some $p_2 \in \|p, \lambda_2; A, B, \lambda_1\|$, $(B, \lambda_1) \downarrow$

$(p', \lambda_2) \cap (B, \lambda_1) \downarrow (p_2, \lambda_2) \neq \emptyset$, but there are no elements of $\|p, \lambda_2; A, B, \lambda_1\|$ for which both hold, and so p' is not an element of $\|p, \lambda_2; A, B, \lambda_1\|$. This is a generalization of the example seen earlier. If such cases occur, the q_S condition will be insufficient; they are eliminated if paths form a densely interlocking network. More precisely:

1) Form the footprints of $\|p, \lambda_2; A, B, \lambda_1\|$ in A and $B : x \equiv (A, \lambda_1) \downarrow (\|p, \lambda_2; A, B, \lambda_1\|, \lambda_2)$ and $y \equiv (B, \lambda_1) \downarrow (\|p, \lambda_2; A, B, \lambda_1\|, \lambda_2)$

2) Close the footprints with all intersecting footprints. That is, define:

$$X_0 \equiv x$$

$$X_{n+1} \equiv \bigcup \{(A, \lambda_1) \downarrow (p', \lambda_2) : (A, \lambda_1) \downarrow (p', \lambda_2) \cap X_n \neq \emptyset\}$$

$$X \equiv \bigcup_{n \in \mathbb{N}} X_n$$

Similarly, starting with y , construct Y .

3) For the q_S condition to be sufficient: For any $p' \notin \|p, \lambda_2; A, B, \lambda_1\|$, if $(A, \lambda_1) \downarrow (p', \lambda_2)$ is in X then $(B, \lambda_1) \downarrow (p', \lambda_2)$ must be disjoint from Y , and if $(B, \lambda_1) \downarrow (p', \lambda_2)$ is in Y then $(A, \lambda_1) \downarrow (p', \lambda_2)$ must be disjoint from X .

With $a = A - X$ and $b = B - Y$, combining $S_{\rightarrow(\lambda_1, A) \rightarrow (\lambda_2, p) \rightarrow}$ and $S_{\rightarrow(\lambda_1, B) \rightarrow (\lambda_2, p) \rightarrow}$ under the q_S condition then has the effect of replacing $\{A, B\}$ with $\{a, b, X \cup Y\}$; the resulting partition will therefore have a total probability of 1. It is interesting to note that statement (3) will be satisfied if the paths are either quite dense or quite sparse; only the intermediate case may cause difficulty.

Combining pairs of outcomes that are determined at multiple λ lead to similar conclusions. Once again, in order for the combination to form a discretely-determined set, the two outcomes can only disagree at a single λ . If the two outcomes disagree on the last measurement then the analysis is little changed from above, except that we need only consider the subset of points in A & B that can be reached from the prior measurements. The trickier case is when a sequence of further measurements come after the two being combined; for example, the case where $S_{\rightarrow(\lambda_1, A) \rightarrow (\lambda_2, C) \rightarrow (\lambda_3, p) \rightarrow}$ and $S_{\rightarrow(\lambda_1, B) \rightarrow (\lambda_2, C) \rightarrow (\lambda_3, p) \rightarrow}$ are combined. To see what the q_S condition demands of the other $p' \in S(\lambda_3)$ in such cases, it will be helpful to expand some the above definitions:

Definition 166. If S is a dynamic set, $\lambda_1 \leq \lambda_2$, $A \subset S(\lambda_1)$, \mathcal{Z} is a set of subsets of $S(\lambda_2)$, and $Z \in \mathcal{Z}$:

$$Z' \in [Z, \mathcal{Z}, \lambda_2; A, \lambda_1] \text{ if } Z' \in \mathcal{Z} \text{ and } (A, \lambda_1) \downarrow (Z', \lambda_2) \cap (A, \lambda_1) \downarrow (Z, \lambda_2) \neq \emptyset$$

If, further, A and B are disjoint subsets of $S(\lambda_0)$:

$$\begin{aligned}
\|Z, \mathcal{Z}, \lambda_2; A, B, \lambda_1\|_0 &\equiv [\lambda_2, \mathcal{Z}, Z; \lambda_1, A] \cap [\lambda_2, \mathcal{Z}, Z; \lambda_1, A] \\
\|Z, \mathcal{Z}, \lambda_2; A, B, \lambda_1\|_{n+1} &\equiv \bigcup_{Z' \in \|Z, \mathcal{Z}, \lambda_2; A, B, \lambda_1\|_n} \|Z', \mathcal{Z}, \lambda_2; A, B, \lambda_1\|_0 \\
\|Z, \mathcal{Z}, \lambda_2; A, B, \lambda_1\| &\equiv \bigcup_{n \in \mathbb{N}^+} \|Z, \mathcal{Z}, \lambda_2; A, B, \lambda_1\|_n
\end{aligned}$$

If for each $p \in S(\lambda_3)$, $Z_p \equiv S_{(\lambda_1, A) \rightarrow (\lambda_3, p)}(\lambda_2)$, and $\mathcal{Z} \equiv \{Z_p : p \in S(\lambda_3)\}$ then $Z_{p'} \in [Z_p, \mathcal{Z}, \lambda_2; A, \lambda_1]$ iff $p' \in [p, \lambda_3; A, \lambda_1]$; from this it follows that $p' \in \|p, \lambda_3; A, B, \lambda_1\|$ iff $Z_{p'} \in \|Z_p, \mathcal{Z}, \lambda_2; A, B, \lambda_1\|$. This allows us to take our earlier analysis on combining $S_{\rightarrow(\lambda_1, A) \rightarrow (\lambda_3, p) \rightarrow}$ and $S_{\rightarrow(\lambda_1, B) \rightarrow (\lambda_3, p) \rightarrow}$, and project it onto any $\lambda_2 \in (\lambda_1, \lambda_3)$: if $S_{\rightarrow(\lambda_1, A) \rightarrow (\lambda_3, p) \rightarrow}$ and $S_{\rightarrow(\lambda_1, B) \rightarrow (\lambda_3, p) \rightarrow}$ are combined, then $S_{\rightarrow(\lambda_1, A) \rightarrow (\lambda_3, p') \rightarrow}$ and $S_{\rightarrow(\lambda_1, B) \rightarrow (\lambda_3, p') \rightarrow}$ must be combined if $Z_{p'} \in \|Z_p, \mathcal{Z}, \lambda_2; A, B, \lambda_1\|$. To extend this to combining $S_{\rightarrow(\lambda_1, A) \rightarrow (\lambda_2, C) \rightarrow (\lambda_3, p) \rightarrow}$ and $S_{\rightarrow(\lambda_1, B) \rightarrow (\lambda_2, C) \rightarrow (\lambda_3, p) \rightarrow}$, simply replace Z_p with $Z_{p, C} \equiv Z_p \cap C$ and \mathcal{Z} with $\mathcal{Z}_C \equiv \{Z_{p, C} : p \in S(\lambda_3)\}$. If $S_{\rightarrow(\lambda_1, A) \rightarrow (\lambda_2, C) \rightarrow (\lambda_3, p) \rightarrow}$ and $S_{\rightarrow(\lambda_1, B) \rightarrow (\lambda_2, C) \rightarrow (\lambda_3, p) \rightarrow}$ are combined then $S_{\rightarrow(\lambda_1, A) \rightarrow (\lambda_2, C) \rightarrow (\lambda_3, p') \rightarrow}$ and $S_{\rightarrow(\lambda_1, B) \rightarrow (\lambda_2, C) \rightarrow (\lambda_3, p') \rightarrow}$ must be combined if $Z_{p', C} \in \|Z_{p, C}, \mathcal{Z}_C, \lambda_2; A, B, \lambda_1\|$. This leads by the same reasoning to the same conclusion: if the network of paths is sufficiently dense (or sparse), the finite and finitely determined partitions that satisfy the q_S condition can be expected to have a total probability of 1.

It follows that if the dynamic set in a dps follows the rule that paths are included unless they can be excluded in principle, then we can reasonably expect the discretely-determined portion of the dps to be maximal.

D. Terminus & Exordium

A word or two is in order about dps's, (X, T, P) , for which X is not a singleton. In such cases $T_{\mathbb{N}}^S$ may be larger than $T_{S\mathbb{N}}$, and so admit partitions that are not nearly-compatible (and therefore not elements of q_S). If the various $(\{S\}, T_S)$ are copies of one another, in the sense of the reductions introduced in Sec. IV B 4, then $T_{\mathbb{N}}^S = T_{S\mathbb{N}}$. For quantum systems, if we consider the various Hilbert space bases as generating the various elements of X , we can partially assert that this is the case. From these bases, each $S(\lambda)$ is equipped with a Σ -algebra, and outcomes are of the form $S_{\rightarrow(\lambda_1, \sigma_1) \rightarrow \dots \rightarrow (\lambda_n, \sigma_n) \rightarrow}$ where the σ 's are elements of their respective Σ -algebra's. For any $S, S' \in X$, $\lambda \in \Lambda$, the Σ -algebra's on $S(\lambda)$ and $S'(\lambda)$ are isomorphic. This gets us part of the way there. One further piece is missing: while the outcomes are isomorphic, the dynamics may not be. In particular, it's possible to have a case where two points, (λ, p_1) and (λ, p_2) , can transition to (λ', p') , but under isomorphisms from

$(S(\lambda), \Sigma)$ to $(S'(\lambda), \Sigma')$ only one of them can. It would then be possible to have $\gamma \in G_{TS'}$, but its isomorphic image not be an element of G_{TS} ; $T_{\mathbb{N}}^S$ may then be larger than $T_{S\mathbb{N}}$. [8]

However, in quantum mechanics we would not expect any such differing dynamics to effect the existence of a reduction between bases; for a given element of q_S we would expect to be able to select sequences of projection operators s.t., under a given basis transformation, the transformed representation will also be an element of q_S . More generally, the discretely determined probabilities ought to be sufficient to determine all inner-products in a given bases, which then entail the discretely determined probabilities in all other bases. Dps's in the various bases are therefore, in a very real sense, copies of each other.

Moreover, it's not clear that all bases are experimentally relevant; naturally, if a basis is not experimentally relevant, then it can't contribute any information to $T_{\mathbb{N}}^S$. Unsurprisingly, given the subject matter of physics, when a basis is given an experimental interpretation, it's generally in terms of paths though space. For example, when the conjugate-coordinate basis is interpreted as momentum, it is necessarily being given a spacial path interpretation; an interpretation predicated on the conjugate-coordinate's mathematical relation to aspects of spacial path probabilities such as the probability flux. This connection between the conjugate-coordinate basis and the particle's spacial path is put on display when the value of the conjugate-coordinate is determined by tracking particle paths in a detector.

However, because quantum systems satisfy all the dps axioms, even if $T_{\mathbb{N}}^S$ is larger than $T_{S\mathbb{N}}$, this fact will have to be reflected in the quantum probabilities. We don't need results like Thm 161 to conclude that quantum systems are dps's, we know that without them. What such results do tell us is that, at least with regard to their measurement theory and a central core of their probability theory, quantum systems appear to be nothing more than fairly generic dps's.

Indeed, one may start to suspect that quantum mechanics, as it currently stands, is a phenomenological theory. It certainly is not holy writ, handed down from on high, which we can't possibly hope to truly understand, but which we must none the less accept in all detail. Rather, it evolved over time by trial and error as a means of calculating results to match newly discovered experimental phenomenon. It has been quite successful in achieving that goal; however it has also historically yielded little understanding of why its calculational methods work. This lack of understanding has often led to claims that we don't understand these matters because we can't understand them; they lie outside the sphere of human

comprehension. While we can not state with certainty that such bold claims of necessary ignorance are false, we can say that they are scientifically unfounded, and so ought to be approached with skepticism. With all humility, it is hoped that this article has lent strength to such skepticism, and shed light on some matters that have heretofore been allowed to lie in darkness.

Appendix A: Δ -Additivity

Quantum probabilities possess an interesting algebraic property that is not a direct consequence of their t -algebra.

For any $t \in T$ s.t. $\{\bigcup t\}$ is also an element of T , define $\Delta(t) \equiv P(t) - P(\{\bigcup t\})$. The Δ function measures the additivity of a gps; if a gps has additive probabilities, then $\Delta(t) = 0$ for all t . Quantum probabilities are not additive, but their Δ function is. For any $t \in T$ s.t. $\{\bigcup t\} \in T$ and for all pairs $\alpha, \beta \in t$, $\{\alpha, \beta\}, \{\alpha \bigcup \beta\} \in T$,

$$\Delta(t) = \sum_{\text{pairs of } \alpha, \beta \in t} \Delta(\{\alpha, \beta\})$$

This is an immediate consequence of quantum probabilities being calculated from expressions of the form $\langle \psi, \lambda_0 | \Pi \Pi^\dagger | \psi, \lambda_0 \rangle$, where Π is a product of projection operators.

Any gps that obeys this rule is Δ -additive.

It follows from the definition of Δ that for disjoint t_1 and t_2 , $\Delta(t_1 \cup t_2) = \Delta(t_1) + \Delta(\{\bigcup t_1\} \cup t_2)$. Applying this to the above formula, we get $\Delta(\{\alpha_1 \cup \alpha_2, \alpha_3\}) = \Delta(\{\alpha_1, \alpha_3\}) + \Delta(\{\alpha_2, \alpha_3\})$, which is directly analogous to the additivity seen in classic probabilities. If t has more than 3 elements then $\Delta(\{\alpha_1 \cup \alpha_2, \alpha_3, \dots\}) = \Delta(\{\alpha_1, \alpha_3, \dots\}) + \Delta(\{\alpha_2, \alpha_3, \dots\}) - \Delta(\{\alpha_3, \dots\})$.

In the orthodox interpretation, Δ -additivity is generally viewed as being due to path interference possessing wave-like properties. An interesting question is, under what conditions will a system obeying the intuitive interpretation display Δ -additivity?

Start by defining a *moment partition* to be any partition, γ , s.t. for any λ , either all $\alpha \in \gamma$ are $bb\lambda$, or all $\alpha \in \gamma$ are $ba\lambda$. (See Defn 147 for the definition of “moment”, and Thm 148 for proof that a partition composed of moments is an ip). Moment partitions have some very useful properties. First, for any $\alpha, \beta \in \gamma$, $(\gamma - \{\alpha, \beta\}) \bigcup (\alpha \bigcup \beta)$ is also a moment partition. Second, if γ is a moment partition, and $\nu \in \gamma$, then $\gamma_\nu \equiv \{\nu, S - \nu\}$ is also a moment partition. Indeed if γ is an ip, and for all $\nu \in \gamma$, $\{\nu, S - \nu\}$ is also an ip, then γ is

a moment partition.

Now take any $\alpha \in t \in T$, any moment partition, γ , and define $\alpha/\gamma \equiv \{\alpha \cap \beta : \beta \in \gamma \text{ and } \alpha \cap \beta \neq \emptyset\}$. $\Delta(\alpha/\gamma)$ is the effect the measurement γ has on the probability that α occurs.

For $\beta \in \alpha/\gamma$ take $P_o(\beta)$ to be the probability that β occurs if all interactions required to measure γ are vanishingly small (while all the interactions for measuring α are unchanged). $P_o(\beta)$ may be thought of as the omniscient probability: we simply know which element of γ occurs without having to perform a measurement. (In the intuitive interpretation, some β occurs even if the measurement does not take place.) It follows that $P(\alpha) = \sum_{\beta \in \alpha/\gamma} P_o(\beta)$. P_o is, of course, simply a conceptual construct; it can only be experimentally determined if $P_o = P$.

Now define $\delta(\beta) \equiv P(\beta) - P_o(\beta)$; $\delta(\beta)$ represents the amount of deflection into/out of β due to the measurement of γ . Since $P(\alpha) = \sum_{\beta \in \alpha/\gamma} P_o(\beta)$, $\Delta(\alpha/\gamma) = \sum_{\beta \in \alpha/\gamma} \delta(\beta)$. Finally, for $\beta \in \alpha/\gamma$, define $\neg\beta \equiv \alpha - \beta$.

Theorem 167. 1) For any outcome, α , any countable moment partition, γ , α/γ is Δ -additive iff $\sum_{\beta \in \alpha/\gamma} \delta(\beta) = \sum_{\beta \in \alpha/\gamma} \delta(\neg\beta)$.

2) A discretely determined dps is Δ -additive iff for every outcome, α , every moment partition, γ , s.t. $\alpha/\gamma \in T$, α/γ is Δ -additive.

Proof. 1) Assume α/γ has $N \geq 3$ elements ($N = 2$ is trivial). Note that since probabilities P_o are fully additive, $(N - 1) \sum_{\beta \in \alpha/\gamma} P_o(\beta) = \sum_{\beta \in \alpha/\gamma} \sum_{\eta \in \alpha/\gamma - \{\beta\}} P_o(\eta) = \sum_{\beta \in \alpha/\gamma} P_o(\neg\beta)$. Adding $(N - 1) \sum_{\beta \in \alpha/\gamma} P_o(\beta)$ to the left side of $\sum_{\beta \in \alpha/\gamma} \delta(\beta) = \sum_{\beta \in \alpha/\gamma} \delta(\neg\beta)$, and $\sum_{\beta \in \alpha/\gamma} P_o(\neg\beta)$ to the right, yields $(N - 2)P(\alpha) + \sum_{\beta \in \alpha/\gamma} P(\beta) = \sum_{\beta \in \alpha/\gamma} P(\neg\beta)$. Now note that $(N - 1) \sum_{\beta \in \alpha/\gamma} P(\beta) = \sum_{\beta \in \alpha/\gamma} P(\alpha/\gamma - \{\beta\})$. Subtracting $(N - 1) \sum_{\beta \in \alpha/\gamma} P(\beta)$ from the left side, and $\sum_{\beta \in \alpha/\gamma} P(\alpha/\gamma - \{\beta\})$ from the left, yields $(N - 2)\Delta(\alpha/\gamma) = \sum_{\beta \in \alpha/\gamma} \Delta(\alpha/\gamma - \{\beta\})$. It remains to show that this is equivalent to $\Delta(\alpha/\gamma) = \sum_{\text{pairs of } \beta, \eta \in \alpha/\gamma} \Delta(\{\beta, \eta\})$.

They are clearly equivalent when $N = 3$. Assume they are equivalent for $N = M$, for $N = M + 1$

$$\begin{aligned} \Delta(\alpha/\gamma) &= \frac{1}{N-2} \sum_{\beta \in \alpha/t} \Delta(\alpha/\gamma - \{\beta\}) \\ &= \frac{1}{N-2} \sum_{\beta \in \alpha/t} \Delta((\alpha - \beta)/\gamma) \\ &= \frac{1}{N-2} \sum_{\beta \in \alpha/\gamma} \sum_{\text{pairs of } \kappa, \eta \in \alpha/\gamma - \{\beta\}} \Delta(\{\kappa, \eta\}) \\ &= \sum_{\text{pairs of } \beta, \eta \in \alpha/\gamma} \Delta(\{\beta, \eta\}) \end{aligned}$$

2) Let's say that T is discretely determined, and for $t \in T$, all $\alpha, \beta \in t$, $\{\alpha \cup \beta\} \in T$. This means that for some $L \subset \Lambda$ s.t t is determined on L , at all elements of L except one, all $\alpha \in T$ correspond to the same outcome.

Take λ to be the element of L at which the various $\alpha \in t$ correspond to different outcomes. Taking $\nu = \bigcup t$, there exists a moment partition at λ , γ , s.t. $t = \nu/\gamma$. \square

So the probabilities of an intuitive system will have the algebraic properties of quantum theory if for all outcomes, α , all countable moment partitions, γ , $\sum_{\beta \in \alpha/\gamma} \delta(\beta) = \sum_{\beta \in \alpha/\gamma} \delta(\neg\beta)$. Let's now see how this requirement may hold.

First note that, since δ measures the effect of the environment on the system, experimental methods should be chosen so as to minimize δ . One way to do this is to perform "passive measurements", a type of measurement that corresponds to how double-slit experiments are generally pictured. In a passive measurement, to determine the probabilities for elements of α/γ , start with some $\gamma_0 \in G_T$ such that $\alpha \in \gamma_0$. Perform the measurement of γ_0 as before, but for $\eta \in \gamma$ block $\neg\eta$ from occurring; if $\neg\eta$ would have occurred, you get a null result. Now if α occurs it means that $\alpha \cap \eta$ occurred. The proportion of α 's to all trials run (including null results) is then $P(\alpha \cap \eta)$. Doing this in turn for each $\eta \in \gamma$ yields the probabilities for α/γ .

Imagine that position measurements on particles are performed in this manner. In the intuitive interpretation, if $\delta(\eta) \neq 0$ it's because the blocking off of $\neg\eta$ creates some minimum but non-vanishing ambient field in η , which deflects the particles. The probabilities on γ_0/γ will sum to 1 if these ambient fields cause the particle's path to be deflected among the outcomes of γ_0 , but not among outcomes of γ . Because γ is a moment partition, this means that paths are not deflected prior to the measurement taking place; this is also a sufficient condition for additivity of final state to hold.

For these passive measurements, $\sum_{\beta \in \alpha/\gamma} \delta(\beta) = \sum_{\beta \in \alpha/\gamma} \delta(\neg\beta)$ if the deflections in $\eta \in \gamma$ due to blocking off $\neg\eta$ are equal to the sum of the deflections in η caused by blocking off each element of $\gamma - \{\eta\}$ individually. So the dps describing particle position will be Δ -additive if the ambient fields in η caused by blocking $\neg\eta$ is equal to the superposition of the ambient fields in η caused by blocking the individual $\nu \in \gamma - \{\eta\}$. There are other ways for Δ -additivity to hold, but this one is conceptually simple, as well as plausible.

Appendix B: Conditional Probabilities & Probability Dynamics

Conditional probabilities are a central concept of probability theory. They are particularly interesting for dps's because probability dynamics are expressed in terms of conditional probabilities.

There are two ways in which a dps's probabilities may be considered to be dynamic. First, the probabilities of what will happen changes as our knowledge of what has happened continues to unfold. Second, given that we measured the system to be in state s at time λ_0 , we may be interested in the probability of measuring the system in state as being in state x at time λ as λ varies. The Schrodinger equation deals with dynamics of this second sort. Conditional probabilities are required for exploring both kinds of dynamics.

The notion of conditional probability is the same in a gps as it is in a classic probability space; essentially, if $P(B) \neq 0$, $P(A|B) = P(A \cap B)/P(B)$. For a dps, the conditional probability of particular interest is the probability that t occurs given that, as of λ , the measurement is consistent with t . In order to delineate this, a few preliminary definitions will prove helpful.

Definition 168. If (X, T, P) is a dps, $t \in T$, and $t \subset \gamma \in G_T$,

$$|t[-\infty, \lambda]|_\gamma \equiv \{|\alpha[-\infty, \lambda]|_\gamma : \alpha \in t\}$$

$$(t)_\lambda^\gamma \equiv \bigcup_{\alpha \in t} (\alpha)_\lambda^\gamma$$

$$(X, T, P) \text{ is } \Lambda\text{-complete if for all } t \in T, \gamma \in G_T \text{ s.t. } t \subset \gamma, \lambda \in \Lambda, (t)_\lambda^\gamma \in T$$

Because $(t)_\lambda^\gamma \subset \gamma$, Λ -completeness places no restrictions on the make-up of G_T , it only places a restriction on the Σ -algebras of the constituent ip probability spaces. It is therefore a fairly weak condition.

$P(t|(t)_\lambda^\gamma)$ is the probability that t occurs given that, as of λ , the measurement is consistent with t . Because probabilities of this type are of interest, it's useful to extend the definition of consistent probabilities (Defn 84) to insure that they are independent of γ .

Definition 169. A dps, (X, T, P) , is *conditionally consistent* if it is Λ -complete and for all $t, t' \in T$, all $\lambda \in \Lambda$, all $t \subset \gamma \in G_T$, $t' \subset \gamma' \in G_T$ s.t. $|t[-\infty, \lambda]|_\gamma = |t'[-\infty, \lambda]|_{\gamma'}$, $P((t)_\lambda^\gamma) = P((t')_\lambda^{\gamma'})$.

Conceptually, this demand is met if probabilities are consistent at all times as the experiments unfold. Note that since $|t[-\infty, \lambda]|_\gamma = |t'[-\infty, \lambda]|_{\gamma'}$, the “all-reet nots” (introduced in

Sec. III E) of $(t)_\lambda^\gamma$ and $(t')_\lambda^{\gamma'}$ are the same; as a result, if a t-algebra is sufficiently rich, it ought to be conditionally consistent.

Definition 170. If (X, T, P) is a conditionally consistent dps, $t, t' \in T$, and $t' \subset \gamma \in G_T$, then with $y \equiv |t'|[-\infty, \lambda]|_\gamma$, $P(t||y) \equiv P(t|(t')_\lambda^\gamma)$.

This is somewhat more intuitive notation for conditional probabilities. The $P(t||y)$ allow us to see how dps probabilities unfold with time. Note that y does not have to be an element of T in order for $P(t||y)$ to be defined.

A common assumption with regard to conditional probabilities on stochastic processes is that they satisfy the Markov property. The equivalent property for dps is:

Definition 171. A dps, (X, T, P) , is *point-Markovian* if for all $t \in T$, $(\lambda, p) \in Uni(\bigcup t)$, if $t = t_{\rightarrow(\lambda, p)} \circ t_{(\lambda, p) \rightarrow}$ then $+t_{\rightarrow(\lambda, p)}, +t_{(\lambda, p) \rightarrow} \in T$ and $P(t) = P(+t_{\rightarrow(\lambda, p)}) \cdot P(+t_{(\lambda, p) \rightarrow})$.

(In the above definition, notation that has been used on sets of dynamic paths have been applied to collections of sets. For example, $t_{\rightarrow(\lambda, p)}$, which is understood to mean $\{\alpha_{\rightarrow(\lambda, p)} : \alpha \in t\}$, and $t_{\rightarrow(\lambda, p)} \circ t_{(\lambda, p) \rightarrow}$, which is understood to mean $\{\alpha_{\rightarrow(\lambda, p)} \circ \beta_{(\lambda, p) \rightarrow} : \alpha, \beta \in t, \alpha_{\rightarrow(\lambda, p)} \circ \beta_{(\lambda, p) \rightarrow} \neq \emptyset\}$. It is hoped that this notation has not caused confusion.)

When the probability function is not additive, this property looses much of its power. None-the-less, the property can generally be assumed to hold, and does have some interesting consequences.

One interesting property of point-Markovian dps's with discrete parameters is that, if the t-algebra is sufficiently rich, then their probabilities tend to be additive. This can be seen for the case of $t \in T$ s.t. for some $\lambda_0 < \lambda$, t is *sbb* λ_0 , *wba* λ , and $(\bigcup t)[\lambda_0, \lambda]$ is finite.[9] Start with any such $t \in T$ and define $t_1 \equiv \{\alpha_{\rightarrow(\lambda-1, q) \rightarrow (\lambda, p) \rightarrow} : \alpha \in t, q \in \alpha(\lambda-1), \text{ and } p \in \alpha(\lambda)\}$; since t is *wba* λ , for any $\gamma \in G_T$ s.t. $t \subset \gamma$, $(\gamma - t) \bigcup t_1$ is an ip, so if it is an element of G_T then $P(t) = P(t_1)$. If, further, the dps is point-Markovian, then $P(\{\alpha_{\rightarrow(\lambda-1, q) \rightarrow (\lambda, p) \rightarrow}\}) = P(\{+\alpha_{\rightarrow(\lambda-1, q)}\}) \cdot P(\{+\alpha_{(\lambda-1, q) \rightarrow (\lambda, p) \rightarrow}\})$. The same manipulations can now be performed on the $+\alpha_{\rightarrow(\lambda-1, q)}$'s. Defining $X_{\bigcup t} \equiv \{+\{\bar{s}[\lambda_0, \lambda]\} : \bar{s} \in \bigcup t\}$, these iterations eventually yield $P(t) = P(X_{\bigcup t})$. Therefore, for any t' s.t. t' is bounded by λ_0 & λ , and $\bigcup t' = \bigcup t$, $P(t') = P(X_{\bigcup t}) = P(t)$.

Appendix C: Invariance On Dynamic Sets and DPS's

1. Invariance On Dynamic Sets

Invariance is among the most fundamental concepts in the study of dynamic systems. We start by establishing the concept for dynamic sets. (In what follows, if f & g are functions, “ $f \cdot g$ ” is the function s.t. $f \cdot g(x) = f(g(x))$)

Definition 172. A *global invariant*, I , on a dynamic set, S , is a pair of functions, $I_\Lambda : \Lambda_S \rightarrow \Lambda_S$ and $I_P : \mathcal{P}_S \rightarrow \mathcal{P}_S$ s.t. $\{I_P \cdot \bar{p} \cdot I_\Lambda : \bar{p} \in S\} = S$.

For any $\bar{p} \in S$, $I(\bar{p}) \equiv I_P \cdot \bar{p} \cdot I_\Lambda$; for $A \subset S$, $I[A] = \{I(\bar{p}) : \bar{p} \in A\}$ (so I is an invariant iff $I[S] = S$).

Global invariants are overly restrictive when Λ is bounded by 0. To handle that case equitably, the following definition of invariance will be employed.

Definition 173. If S is a dynamic set and Λ_S is unbounded from below, then I is an *invariant* on S if it is a global invariant on S .

If Λ_S is bounded from below then I is an *invariant* on S if there exists a dynamic set, S^* s.t. Λ_{S^*} is unbounded from below, $S = S^*[0, \infty]$, and I is an invariant on S^* .

Unless stated otherwise, for the remainder of this section dynamic sets will be assumed to be unbounded from below.

Theorem 174. If I is an invariant on S

- 1) I_P is a surjection
- 2) I_Λ and I_P are invertible iff I_Λ is an injection and I_P is a bijection
- 3) I_Λ and I_P are invertible and $I^{-1} \equiv (I_\Lambda^{-1}, I_P^{-1})$ is an invariant on S iff I_Λ and I_P are bijections

Proof. 1) If it is not then $\mathcal{P}_{I[S]} \subset \text{Ran}(I_P) \subsetneq \mathcal{P}_S$ and so $I[S] \neq S$.

2) Follows from (1) and that functions are invertible iff they are injections.

3) \Rightarrow Follows from (2) and that for I^{-1} to be an invariant, the domain of I_Λ^{-1} needs to be Λ .

\Leftarrow Because I_Λ and I_P are bijections, for all $\bar{p} \in S$ $I^{-1}(I(\bar{p})) = I_P^{-1} \cdot I_P \cdot \bar{p} \cdot I_\Lambda \cdot I_\Lambda^{-1} = \bar{p}$. Therefore $I^{-1}[I[S]] = S$. Since $I[S] = S$, $I^{-1}[S] = S$. \square

Definition 175. Dynamic set S is *weakly Λ -invariant* if for all $\lambda \in \Lambda$ there's an invariant, L^λ , s.t. for all $\lambda' \in \Lambda_S$, $L^\lambda_\Lambda(\lambda') = \lambda' + \lambda$.

S is *strongly Λ -invariant* if it is weakly Λ -invariant and for all $\lambda \in \Lambda$, $L^\lambda_{\mathcal{P}}$ is the identity on \mathcal{P}_S .

Theorem 176. *A dynamic space D is strongly Λ -invariant iff it is homogeneous and all of \mathcal{P}_S is homogeneously realized*

Proof. When Λ_D is unbounded from below, this is fairly clear.

If Λ_D is bounded from below:

\Rightarrow follows immediately from the unbounded case.

For \Leftarrow , it's necessary to construct a homogeneous, homogeneously realized, unbounded from below D^* s.t. $D^*[0, \lambda] = D$. This is relatively easy. Note that if D is homogeneous & homogeneously realized then for any $\Delta\lambda > 0$, any $\lambda, \lambda' \in \Lambda_D$, $D[\lambda, \lambda + \Delta\lambda]$ and $D[\lambda', \lambda' + \Delta\lambda]$ are copies of each other, in that if you move $D[\lambda', \lambda' + \Delta\lambda]$ to λ , the two are equal. So to construct D^* , take any $D[0, \Delta\lambda]$, append it to the beginning of D , then append it to the beginning of the resulting set, and so forth. Only one set, D^* , will equal this construction for all intervals, $[-n\Delta\lambda, \infty]$; D^* is a homogeneous, homogeneously dynamic space, and Λ_{D^*} is unbounded from below, so D^* is strongly Λ -invariant. \square

Definition 177. S is *reversible* if for all $\lambda \in \Lambda$ there's an invariant, R^λ , s.t. for all $\lambda' \in \Lambda_S$, $R^\lambda_\Lambda(\lambda') = \lambda - \lambda'$, and $R^\lambda_{\mathcal{P}} \cdot R^\lambda_{\mathcal{P}}$ is the identity on \mathcal{P}_S .

Theorem 178. 1) *If S is reversible then it is weakly Λ -invariant.*

2) *If S is reversible and for all $\lambda_1, \lambda_2 \in \Lambda$, $R^{\lambda_1}_{\mathcal{P}} = R^{\lambda_2}_{\mathcal{P}}$ then it is strongly Λ -invariant.*

Proof. 1) For every $\lambda \in \Lambda_S$, $L^\lambda = R^\lambda \cdot R^0$ is an invariant (because the composite of any two invariants is an invariant; R^0 being R^λ with $\lambda = 0$), and for all $\lambda' \in \Lambda$, $L^\lambda_\Lambda(\lambda') = R^\lambda_\Lambda(R^0_\Lambda(\lambda')) = \lambda' + \lambda$.

2) Again with $L^\lambda = R^\lambda \cdot R^0$, if for all $\lambda_1, \lambda_2 \in \Lambda$, $R^{\lambda_1}_{\mathcal{P}} = R^{\lambda_2}_{\mathcal{P}}$ then $R^\lambda_{\mathcal{P}} \cdot R^0_{\mathcal{P}} = R^\lambda_{\mathcal{P}} \cdot R^0_{\mathcal{P}}$, which is the identity on \mathcal{P}_S . \square

2. Invariance on DPS's

To apply the notion of invariance to dps's, we'll start by expanding the definition to cover invariance on a collection of dynamic sets. (In the interest of concision, previous

considerations for the case where Λ is bounded by 0 will not be explicitly mentioned, but they should be assumed to continue to apply.)

Definition 179. If X is a collection of dynamic sets, I is an *invariant* on X if it is a pair of functions $I_\Lambda : \bigcup_{S \in X} \Lambda_S \rightarrow \bigcup_{S \in X} \Lambda_S$ and $I_\mathcal{P} : \bigcup_{S \in X} \mathcal{P}_S \rightarrow \bigcup_{S \in X} \mathcal{P}_S$ s.t. for some bijection, $B_I : X \rightarrow X$, all $S \in X$, $I[S] \equiv \{\bar{p} : \text{for some } \bar{p}' \in S, \bar{p} = I_\mathcal{P} \cdot \bar{p}' \cdot I_\Lambda\} = B_I(S)$.

In the case where $X = \{S\}$ this reduces to the prior definition of invariance. When B_I is simply the identity on X , I represents a collection of dynamic set invariants, one for each element of X . In the more general case, where the dynamic sets in the collection are allowed to map onto each other under the transformation, the collection can be invariant under the transformation even when none of the individual dynamic sets are.

Generalizing types of invariance is straightforward. For example:

Definition 180. X is *reversible* if for all $S, S' \in X$, $\Lambda_S = \Lambda_{S'} = \Lambda$, and for all $\lambda \in \Lambda$ there's and invariant on X , R^λ , s.t. for each $S \in X$, $R^\lambda_\mathcal{P} \cdot R^\lambda_\Lambda$ is the identity on \mathcal{P}_S , and for all $\lambda' \in \Lambda$, $R^\lambda_\Lambda(\lambda') = \lambda - \lambda'$.

Invariance on a dps is now:

Definition 181. If (X, T, P) is a dps, and I is an invariant on X , I is an *invariant* on (X, T, P) if

- 1) If $\gamma \in G_T$ and $I[\gamma]$ is an ip on some $S \in X$ then $I[\gamma] \in G_T$
- 2) If $t \in T$ and for some $\gamma \in G_T$, $I[t] \subset \gamma$ then $I[t] \in T$
- 3) If $t \in T$ and $I[t] \in T$ then $P(t) = P(I[t])$

The dps versions of invariants such as reversibility follow immediately.

Appendix D: Parameter Theory

Definition 182. A set, Λ , together with a binary relation on Λ , $<$, a binary function on Λ , $+$, and a constant $0 \in \Lambda$, is an *open parameter* if the structure, $(\Lambda, <, +, 0)$, satisfies the following:

Total Ordering:

- 1) For all $\lambda \in \Lambda$, $\lambda \not< \lambda$
- 2) For all $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$, if $\lambda_1 < \lambda_2$ and $\lambda_2 < \lambda_3$ then $\lambda_1 < \lambda_3$

3) For all $\lambda_1, \lambda_2 \in \Lambda$, either $\lambda_1 = \lambda_2$ or $\lambda_1 < \lambda_2$ or $\lambda_2 < \lambda_1$

Addition:

4) For all $\lambda \in \Lambda$, $\lambda + 0 = \lambda$

5) For all $\lambda_1, \lambda_2 \in \Lambda$, $\lambda_1 + \lambda_2 = \lambda_2 + \lambda_1$

6) For all $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$, $(\lambda_1 + \lambda_2) + \lambda_3 = \lambda_1 + (\lambda_2 + \lambda_3)$

The standard interrelationship between ordering and addition:

7) For all $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$, $\lambda_1 < \lambda_2$ iff $\lambda_1 + \lambda_3 < \lambda_2 + \lambda_3$

8) For all $\lambda_1, \lambda_2 \in \Lambda$, if $\lambda_1 < \lambda_2$ then there's a $\lambda_3 \in \Lambda$ s.t. $\lambda_1 + \lambda_3 = \lambda_2$

Possesses a positive element:

9) There exists a $\lambda \in \Lambda$ s.t. $\lambda > 0$

The enumerated statements in this definition will be referred to as the “open parameter axioms”, and the individual statements will be referred to by number: opa 1 referring to “For all $\lambda \in \Lambda$, $\lambda \not< \lambda$ ”, etc.

Because in most cases $<$, $+$, and 0 will be immediately apparent given the set Λ , parameters will often be referred to by simply referring to Λ .

Definition 183. For $\lambda \in \Lambda$, $\lambda' \in \Lambda$ is an *immediate successor* to λ if $\lambda' > \lambda$ and there does not exist a $\lambda'' \in \Lambda$ s.t. $\lambda' > \lambda'' > \lambda$.

A parameter is *discrete* if every $\lambda \in \Lambda$ has an immediate successor.

A parameter is *dense* if no $\lambda \in \Lambda$ has an immediate successor.

Theorem 184. *An open parameter is either discrete or dense*

Proof. Follows from opa 8 and opa 7. □

Definition 185. For a discrete open parameter, the immediate successor to 0 is $\mathbf{1}$.

Theorem 186. *If Λ is discrete, then for every $\lambda \in \Lambda$, the immediate successor to λ is $\lambda + \mathbf{1}$.*

Proof. This too follows from opa 8 and opa 7. □

Definition 187. If $\chi \subset \Lambda$, $\lambda \in \Lambda$ is an *upper-bound* of χ if for every $\lambda' \in \chi$, $\lambda \geq \lambda'$; λ is the *least upper-bound* if it is an upper-bound, and given any other upper-bound, λ_+ , $\lambda \leq \lambda_+$. If χ has no upper-bound, then χ is *unbounded from above*.

Similarly, $\lambda \in \Lambda$ is an *lower-bound* of χ if for every $\lambda' \in \chi$, $\lambda \leq \lambda'$; λ is the *greatest lower-bound* if it is a lower-bound, and given any other lower-bound, $\lambda_-, \lambda_- \leq \lambda$. If χ has no lower-bound, then χ is *unbounded from below*.

λ' is the *additive inverse* of λ if $\lambda + \lambda' = 0$

Theorem 188. *If Λ is an open parameter*

- 1) *Λ is unbounded from above.*
- 2) *If Λ is bounded from below, it's greatest lower bound is 0.*
- 3) *If Λ does not have a least element, every $\lambda \in \Lambda$ has an additive inverse; if it does have a least element, only 0 has an additive inverse*

Proof. 1) Follows from opa 9 and opa 7 (with help from opa 4)

2) Also follows from opa 7 with help from opa 4

3) First take the case where Λ does not have a least element. If $\lambda < 0$ then by opa 8 there's a λ' s.t. $\lambda + \lambda' = 0$. If $\lambda > 0$ take any λ' s.t. $\lambda + \lambda' < 0$ (such a λ' must exist because Λ is unbounded from below). As just established, there must be a $\Delta\lambda$ s.t. $\lambda + \lambda' + \Delta\lambda = 0$, so $\lambda' + \Delta\lambda$ is the additive inverse of λ .

Now assume Λ does have a least element. By opa 7 and pt. 2 of this theorem, for any $\lambda, \lambda' \in \Lambda$ s.t. $\lambda \neq 0$, $\lambda + \lambda' \geq \lambda > 0$. □

Open parameters admit models which, under most interpretations of “parameter”, would not be considered admissible. For example, infinite ordinals (with the expected interpretations of $<$, $+$, and 0) are open parameters, as are the extended reals. Rational numbers are also open parameters, as are numbers which, in decimal notation, have only a finite number of non-zero digits.

To eliminate these less-than-standard models, parameters will be defined as open parameters that are finite (sometimes called Archimedean), and either discrete or continuous (that is, parameters have the crucial property that all limits which tend toward fixed, finite values exist). This will be accomplished through the well known method of adding a “completeness axiom”.

Definition 189. An open parameter, Λ , is a *parameter* if every non-empty subset of Λ that is bounded from above has a least upper bound.

From here on out, it will be assumed that “ Λ ” refers to a parameter.

Definition 190. λ added to itself n times will be denoted $n\lambda$ (for example $3\lambda \equiv \lambda + \lambda + \lambda$).
 $0\lambda \equiv 0$.

$$\lambda - \lambda' \equiv \begin{cases} 0 & \text{if } \Lambda \text{ is bounded by } 0 \text{ and } \lambda < \lambda' \\ \Delta\lambda & \text{where } \Delta\lambda + \lambda = \lambda', \text{ otherwise} \end{cases}$$

Theorem 191. If Λ is a parameter then for every $\lambda_1, \lambda_2 \in \Lambda$, $0 < \lambda_1 < \lambda_2$, there exists an $n \in \mathbb{N}$ s.t. $n\lambda_1 > \lambda_2$.

Proof. Assume that for all $n \in \mathbb{N}$, $n\lambda_1 < \lambda_2$. Then the set $\chi = \{x \in \Lambda : x = n\lambda_1\}$, is bounded from above. Therefore it has a least upper bound, λ_M . Since $\lambda_1 > 0$, $\lambda_M - \lambda_1 < \lambda_M$, and so $\lambda_M - \lambda_1$ can't be an upper bound of χ . Therefore for some $i \in \mathbb{N}$, $i\lambda_1 > \lambda_M - \lambda_1$. But then $(i+2)\lambda_1 > \lambda_M$, so λ_M can not be an upper bound. Therefore χ is unbounded, and so for some $n \in \mathbb{N}$, $n\lambda_1 > \lambda_2$. \square

Thm 191 is equivalent to saying that for all $\lambda \in \Lambda$, λ is finite.

Theorem 192. For any $n \in \mathbb{N}^+$

- 1) $\lambda_1 > \lambda_2$ iff $n\lambda_1 > n\lambda_2$
- 2) $\lambda_1 = \lambda_2$ iff $n\lambda_1 = n\lambda_2$

Proof. 1) Follows from opa 7, opa 2

2) Follows from (1) and opa 3. \square

Theorem 193. If Λ is a discrete parameter then given any $\lambda \in \Lambda$, $\lambda > 0$ there's an $n \in \mathbb{N}$ s.t. $\lambda = n\mathbf{1}$.

Proof. Follows from Thms 186 and 191 \square

Theorem 194. For any $n, m \in \mathbb{N}$

- 1) $n\mathbf{1} + m\mathbf{1} = (n+m)\mathbf{1}$
- 2) If $n > m$ then $n\mathbf{1} > m\mathbf{1}$

Proof. 1) Follows from opa 6

2) Take $k = n - m$; By (1) and opa 7, (2) holds iff $k\mathbf{1} > 0$, which follows from Thm 192.1. \square

Thms 193 and 194, together with Thm 188.3, create a complete characterization of discrete parameters. A similar characterization for dense parameters will now be sketched.

Definition 195. For $\lambda, \lambda' \in \Lambda$, $m \in \mathbb{N}$, $n \in \mathbb{N}^+$, $\lambda' = \frac{m}{n}\lambda$ if $n\lambda' = m\lambda$.

Theorem 196. If Λ is dense then for all $\lambda \in \Lambda$, $m \in \mathbb{N}$, $n \in \mathbb{N}^+$

- 1) $\frac{m}{n}\lambda \in \Lambda$
- 2) If $\lambda' = \frac{m}{n}\lambda$ and $\lambda'' = \frac{m}{n}\lambda$ then $\lambda' = \lambda''$

Proof. 1) Take $\chi \equiv \{\lambda' \in \Lambda : n\lambda' \leq m\lambda\}$. χ is bounded from above, so take λ_χ to be the least upper bound. If $n\lambda_\chi$ is either greater or less than $m\lambda$ then density provides an example that contradicts λ_χ being the least upper bound, so by opa 3 $n\lambda_\chi = m\lambda$.

- 2) $n\lambda' = m\lambda$ and $n\lambda'' = m\lambda$, so $\lambda' = \lambda''$ by Thm 192.2. □

Theorem 197. If Λ is dense then for all $\lambda \in \Lambda$

- 1) If $m_1, m_2 \in \mathbb{N}$, $n_1, n_2 \in \mathbb{N}^+$ and $\frac{m_1}{n_1} = \frac{m_2}{n_2}$ then $\frac{m_1}{n_1}\lambda = \frac{m_2}{n_2}\lambda$
- 2) For $q_1, q_2 \in \mathbb{Q}$, $q_1\lambda + q_2\lambda = (q_1 + q_2)\lambda$
- 3) For $q_1, q_2 \in \mathbb{Q}$, $\lambda > 0$, if $q_1 > q_2$ then $q_1\lambda > q_2\lambda$

Proof. 1) If $\frac{m_1}{n_1} = \frac{m_2}{n_2}$ then either there exist a m, n, k_1, k_2 s.t. $m_1 = k_1m$, $n_1 = k_1n$, $m_2 = k_2m$, and $n_2 = k_2n$. It is sufficient to show that $\frac{m_1}{n_1}\lambda = \frac{m_2}{n_2}\lambda$. Taking $\lambda' \equiv \frac{m_1}{n_1}\lambda$, $(k_1n)\lambda' = (k_1m)\lambda$. By Thm 194.1, $k_1(n\lambda') = k_1(m\lambda)$. By Thm 192.2 $n\lambda' = m\lambda$.

2) For some $m_1, m_2 \in \mathbb{N}$, $n \in \mathbb{N}^+$, $q_1 = \frac{m_1}{n}$ and $q_2 = \frac{m_2}{n}$. With $\lambda_1 \equiv \frac{m_1}{n}\lambda$ and $\lambda_2 \equiv \frac{m_2}{n}\lambda$, $m_1\lambda + m_2\lambda = n\lambda_1 + n\lambda_2$ and so by opa 6, $n(\lambda_1 + \lambda_2) = (m_1 + m_2)\lambda$, which mean $\lambda_1 + \lambda_2 = \frac{(m_1+m_2)}{n}\lambda$.

3) Follows from (2) and opa 7, and the fact that $q_3 \equiv q_2 - q_1$ is a positive rational number (note that if $\lambda > 0$ and $q > 0$ then $q\lambda > 0$). □

Definition 198. A parameter sequence, $(\lambda_n)_{n \in \mathbb{N}}$, is *convergent* if there exists a $\lambda' \in \Lambda$ s.t. for any $\Delta\lambda > 0$ there's an $n \in \mathbb{N}$ s.t. for all $i > n$, $\lambda_i \in (\lambda' - \Delta\lambda, \lambda' + \Delta\lambda)$. In this case we say $\lambda = \lim \lambda_n$.

$(\lambda_n)_{n \in \mathbb{N}}$, is *Cauchy-convergent* if for any $\Delta\lambda > 0$ there's an $n \in \mathbb{N}$ s.t. for all $i, j > n$, $\lambda_j \in (\lambda_i - \Delta\lambda, \lambda_i + \Delta\lambda)$.

$(\lambda_n)_{n \in \mathbb{N}}$ is *monotonic* if either for all an $i \in \mathbb{N}$, $\lambda_{i+1} \geq \lambda_i$, or for all an $i \in \mathbb{N}$, $\lambda_{i+1} \leq \lambda_i$.

Theorem 199. If Λ is a dense parameter and $(q_n)_{n \in \mathbb{N}}$ is a monotonic, Cauchy-convergent sequence of rational numbers, then $(q_n\lambda)_{n \in \mathbb{N}}$ is a convergent parameter sequence (with the understanding that if Λ is bounded from below then all q_i are non-negative).

Proof. A: If $(q_n)_{n \in \mathbb{N}}$ is a Cauchy-convergent sequence of rational numbers, then $(q_n \lambda)_{n \in \mathbb{N}}$ is a Cauchy-convergent.

- Follows from Thm 197 and the fact that the set of rational numbers is dense. -

B: If Λ is a parameter, and $A \subset \Lambda$ is bounded from below, then A must have a greatest lower bound

- Take B to be the set of lower bounds of A . B is bounded from above by every element of A , so take b to be the least upper bound of B . b must be a lower bound of A because if for any $a \in A$, $a < b$ then b can not be the least upper bound of B . It also must be the greatest lower bound, because if any $x > b$ is a lower bound of A , then $x \in B$, in which case b would not be an upper bound of B . -

Assume $(q_n)_{n \in \mathbb{N}}$ is monotonically increasing. Take λ' to be the least upper bound of $\text{Ran}((q_n \lambda)_{n \in \mathbb{N}})$. Take any $\Delta \lambda > 0$. By (A) there exists an $n \in \mathbb{N}$ s.t. for all $i, j > n$, $q_j \lambda \in (q_i \lambda - \Delta \lambda, q_i \lambda + \Delta \lambda)$. It follows that for all $i > n$, $q_i \lambda \in (\lambda' - \Delta \lambda, \lambda' + \Delta \lambda)$.

By (B), the proof for monotonically decreasing sequences is similar. \square

It is a foundational result of real analysis that all Cauchy-convergent sequences of rational numbers converge to a real number, and for all real numbers there exist monotonic, Cauchy-convergent sequences of rational numbers that converge to it.

Theorem 200. *If $(q_n)_{n \in \mathbb{N}}$ and $(q'_n)_{n \in \mathbb{N}}$ are two monotonic sequences of rational numbers that converge to the same real number then $\lim(q_n \lambda) = \lim(q'_n \lambda)$.*

Proof. Assume $(q_n)_{n \in \mathbb{N}}$ and $(q'_n)_{n \in \mathbb{N}}$ are monotonically increasing. Because they converge to the same real number, they have the same least upper bound. $(q_n \lambda)_{n \in \mathbb{N}}$ and $(q'_n \lambda)_{n \in \mathbb{N}}$ must then also have the same least upper bound. By the proof of Thm 199, they have the same limit.

All other cases are similar. \square

Definition 201. If $(q_n)_{n \in \mathbb{N}}$ is a monotonic, Cauchy-convergent sequence of rational numbers and $\lim q_n = r$ then for any $\lambda \in \Lambda$, $r \lambda \equiv \lim(q_n \lambda)$.

By Thm 200 the above definition uniquely defines multiplication by a real number.

Theorem 202. *If Λ is a dense parameter and $\mathbf{1}$ any element of Λ that greater than 0*

- 1) *For any real number r , $r \mathbf{1} \in \Lambda$*
- 2) *For any $\lambda \in \Lambda$, there exists a real number, r , s.t. $r \mathbf{1} = \lambda$*

Proof. 1) Follow immediately from Thm 199.

2) We'll take the case of $\lambda > 0$; $\lambda < 0$ is similar and $\lambda = 0$ is trivial.

A: If $\lambda > 0$ then there exists a rational number q s.t $0 < q\mathbf{1} < \lambda$

- The greatest lower bound of the set of $\frac{1}{2^n}\mathbf{1}$, $n \in \mathbb{N}^+$, is 0; since $\lambda > 0$ there must be an i s.t. $\frac{1}{2^i}\mathbf{1} < \lambda$ -

Take q to be any rational number s.t. $q\mathbf{1} < \lambda$. By Thm 191 there's an $n \in \mathbb{N}^+$ s.t. $nq\mathbf{1} > \lambda$; take m_0 to be the smallest such element of \mathbb{N}^+ . Take $\lambda_0 = (m_0 - 1)q\mathbf{1}$; note that $\lambda_0 \leq \lambda$ and $\lambda - \lambda_0 \leq q\mathbf{1}$. Similarly for each $i \in \mathbb{N}$ take m_i to be the smallest element of \mathbb{N}^+ s.t. $m_i(\frac{q}{2^i})\mathbf{1} > \lambda$. With $\lambda_i \equiv (m_i - 1)(\frac{q}{2^i})\mathbf{1}$ it follows that $\lambda_i \leq \lambda_{i+1} \leq \lambda$ and $\lambda_i - \lambda \leq (\frac{q}{2^i})\mathbf{1}$.

Consider the sequence $((m_i - 1)(\frac{q}{2^i})\mathbf{1})_{n \in \mathbb{N}}$; for any $\Delta\lambda$ there's an n s.t. $(\frac{q}{2^n}) < \Delta\lambda$. For all $j > n$, $\lambda - (m_j - 1)(\frac{q}{2^j})\mathbf{1} < (\frac{q}{2^n})\mathbf{1} < \Delta\lambda$, so $((m_i - 1)(\frac{q}{2^i})\mathbf{1})_{n \in \mathbb{N}}$ converges to λ . with $r = \lim(m_i - 1)(\frac{q}{2^i})$, $\lambda = r\mathbf{1}$. \square

Theorem 203. *If Λ is a dense parameter and $\mathbf{1}$ any element of Λ that greater than 0*

1) For $r_1, r_2 \in \mathbb{R}$, $r_1\lambda + r_2\lambda = (r_1 + r_2)\lambda$

2) For $r_1, r_2 \in \mathbb{R}$, $\lambda > 0$, if $r_1 > r_2$ then $r_1\lambda > r_2\lambda$

Proof. 1) If $(q_n)_{n \in \mathbb{N}}$ and $(q'_n)_{n \in \mathbb{N}}$ are monotonically increasing sequences of rational numbers and $\lim q_n = r_1$ and $\lim q'_n = r_2$ then $\lim(q_n + q'_n) = r_1 + r_2$.

2) Follows from (1) and opa 7, and the fact that $r_3 \equiv r_2 - r_1$ is a positive real number. \square

Thms 202 & 203 create a complete characterization of dense parameters.

[1] It should be pointed that, while the analogy between experiments and automata is useful for creating a quick sketch of the theory, it is no more than an analogy. In the automata studied by computer scientists, time is assumed to be discrete and the number of automata states is assumed to be finite. These assumptions must be made in order to assert that the automata are performing “calculations”, and they have significant impact in deriving classes of calculable functions, but such assumptions would be out of place when discussing experiments.

[2] The mathematical notion of a “model” is a basic concept from the field of mathematical logic. Since mathematical logic is not generally a part of the scientific curriculum, here's a brief description of model theory. A mathematical theory is a set of formal statements, generally

taken to be closed under logical implication. A model for the theory is a “world” in which all the statements are true. For example, group theory starts with three statements involving a binary function, \cdot , and a constant, I . They are: For all x, y, z , $(x \cdot y) \cdot z = x \cdot (y \cdot z)$; For all x , $x \cdot I = x$; For all x there exists a y s.t. $x \cdot y = I$. One model for this theory is the set of integers, with “ \cdot ” meaning $+$ and “ I ” meaning 0. Another is the set of non-zero real numbers, with “ \cdot ” meaning \times and “ I ” meaning 1. It is generally the case that a theory will have a more than one model. A model may be thought of as a reality that underlies the theory (in which case the fact that a theory has many different models simply means that it holds under many different circumstances). It is this sense of models referring to a theory’s underlying reality that leads to the correspondence between scientific interpretations and mathematical models.

- [3] Had any part of \mathcal{P}_S needed to be homogeneously realized it could have presented a problem with viewing S as playing out on the stage of space-time, because every point in space-time can only be realized at a single λ .
- [4] In the Introduction two models were for non-determinism were given. One of them, type-m non-determinism, encounters a well known difficulty at this point: in order for the statistical view of probabilities to be applicable, every individual run of an experiment must result in an individual outcome being obtained. However, if an e-automata displays pure type-m non-determinism, it will simultaneously take all paths for all possible outcomes, not just paths which cross some particular $[e]$, and this will result in multiple outcomes. In this controversy, experimental results have decided in favor of individual outcomes; experimental apparatus always end up in a single state, and as outcomes have been defined, each individual final environmental state corresponds to an individual outcome.
- [5] (X, T_c, P_c) does, however, meet all the other requirements for being a gps; that is, if for all pairs of c-sets, A and B , $(\bigcup A) \cap (\bigcup B) \in T_c$ then (X, T_c, P_c) is a gps
- [6] If $\langle \lambda_1, s_1 | \lambda_2, s_2 \rangle \neq 0$, then it’s clear that $(\lambda_1, s_1) \Rightarrow (\lambda_2, s_2)$. However, there is some ambiguity as to whether $\langle \lambda_1, s_1 | \lambda_2, s_2 \rangle = 0$ necessarily implies that the transition can not take place, or if it simply demands that the transition occurs with probability 0. This question grows acute if there’s a (λ, s) s.t. $\lambda_1 < \lambda < \lambda_2$, $\langle \lambda_1, s_1 | \lambda, s \rangle \neq 0$, and $\langle \lambda, s | \lambda_2, s_2 \rangle \neq 0$. In that case, in order for (5) to hold, we would have to allow the transition $(\lambda_1, s_1) \Rightarrow (\lambda_2, s_2)$, but say that it occurs with probability 0. There’s no necessity to define the \Rightarrow relation in this manner, but doing so is in keeping both with the path integral formalism and the orthodox interpretations

of quantum mechanics. There one would say that there exist possible paths from (λ_1, s_1) to (λ_2, s_2) , but they interfere with each other in such a way as to keep the total amplitude of the transition 0; however, if a further measurement caused only a subset of these paths to be taken, then the probability can become non-zero.

- [7] The $\sim |\alpha|_\lambda^+$ encountered in Section III E are examples of measurements of rate of change; the $\sim |\alpha|_\lambda^+$ contain paths that are in some compatible set, t , as of λ , but whose “velocities” ensure that they will exit t immediately after λ
- [8] Then again it may not. For example, if the transitions are different because the states are deterministic, or conserved, in S_1 but not in S_2 , this will not effect the existence of a reduction. That is because, when states are deterministic, the order in which measurements occur is irrelevant, so many different measurement sequences will result in the same ip.
- [9] Finite spin systems are point-Markovian, and all their $t \in T$ satisfy these restrictions; however their parameters are generally assumed to be non-discrete, which does allow them to have non-additive probabilities.